LECTURES ON THE THEORY OF PC SEQUENCES

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Content

Dear Harry

Thank you for introduction me to PC Sequences.
Thank you for teaching me PC Sequences.
Thank you for several hours of discussion.
Thank you for being a friend to me.

Stochastic Sequence

A *stochastic sequence (SS)* is a sequence $(x(n))$, $n \in Z$, of elements in a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$.
The *auto-covariance function* of $(x(n))$:

$$R_x(m,n) = (x(m), x(n)) \qquad m, n \in Z.$$

Two sequences with the same auto-covariance function are identified (unitary equivalent), $(x(n)) \approx (y(n))$.

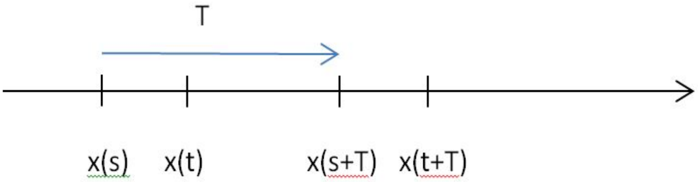
For example

$$(i.i.d.N(0,1)) \approx (\text{any orthonormal basis}) \approx ((1/\sqrt{2\pi})e^{in\cdot}) \subseteq L^2(\mathcal{C})$$

PC Sequences

A periodically correlated sequence with period $T > 0$, $T \in \mathcal{Z}$, (T-PC) is a SS $(x(n))$ such that

$$R_x(m,n) = R_x(m+T, n+T), \quad m,n \in Z.$$



Important parameter of a PC sequence is the sequence

$$a_j(n) := \sum_{r=0}^{T-1} e^{-2\pi ijr/T} R_x(n+r, r), \qquad j = 0, \ldots, T-1.$$

Structure of PC Sequences

Theorem (Makagon, Miamee, 2014)

$(x(n))$ in $\mathcal{H}$ is T-PC iff there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, $x \in \mathcal{K}$, and two unitary operators U, V in $\mathcal{K}$ such that

$$x(n) = \frac{1}{T} \sum_{j=0}^{T-1} e^{-2\pi ijn/T} U^n V^j x, \quad n \in Z, \tag{1}$$

where

- $V^T = I$
- $V^j U^n = e^{-2\pi itj/T} U^n V^j$ *(CCR condition)*

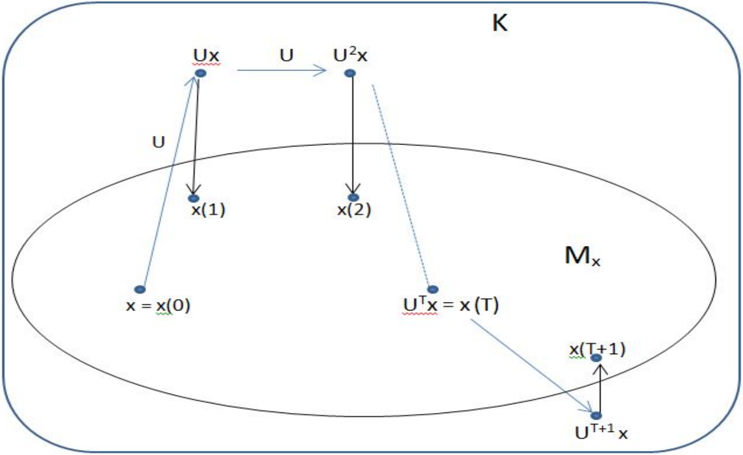
If $\mathcal{K} = \overline{span}\{U^n V^j x : t, j \in Z\}$, then $(\mathcal{K}, U, V, x)$ are uniquely determined by $(x(t))$ in the sense of unitary equivalence.

Obvious Corollaries

Since $V^j U^n = e^{-2\pi itj/T} U^n V^j$ and $V^T = I$

$$
\begin{aligned}
x(n) &= \frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} \underbrace{\left[U^n V^j x\right]}_{X^j(n)} \\
&= \underbrace{U^n \left[\frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} V^j x\right]}_{p(n)} \\
&= \underbrace{\left[\frac{1}{T}\sum_{j=0}^{T-1} V^j\right]}_{P_0} (U^n x)
\end{aligned}
$$

[Gladyshev, Hurd]


K
Ux
U
U²x
U
x(1)
x(2)
Mx
x = x(0)
UTx = x (T)
x(T+1)
UT+1 x

Unitary representation of a group

Theorem (SNAG Theorem: Stone, Naimark, Ambrose, Godement)

If T^g is a continuous unitary representation of an LCA group G in $\mathcal{H}$ (i.e. $T^{g+h} = T^g T^h$), then there exists a unique orthogonal projection-valued measure $E(\cdot)$ on $\hat{G}$ such that

$$T^g = \int_{\hat{G}} \gamma(g) E(d\gamma), \quad g \in G$$

U^n and V^j are representations of Z and $Z_T = \{0, 1, \ldots, T-1\}$:

$$U^n = \int_0^{2\pi} e^{-iun} E(du), \qquad V^j = \sum_{k=0}^{n-1} e^{-2\pi ikj/T} P_k$$

Harmonizability

Recall $x(n) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} U^n V^j x.$

1. $$x(n) = \int_0^{2\pi} e^{-iun} Z(du)$$

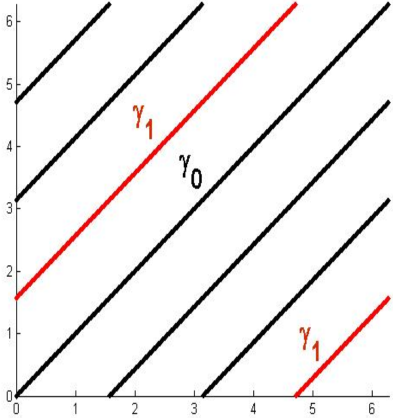
   Proof: Define $Z(du) = \frac{1}{n}\sum_{j=0}^{n-1} E(du - 2\pi j/n) V^j x$ $\square$

2. $$R(m,n) = \int_0^{2\pi}\int_0^{2\pi} e^{-i(mu-nv)} \Gamma(du, dv)$$

   Proof: Define $\Gamma(du, dv) = (Z(du), Z(dv))$ $\square$

[Gladyshev, Hurd]

Measure Γ



Spectrum of PC Sequence

Theorem (Hurd)

If $(x(n))$ is PC, then there are measures γ_j, $j = 0, \ldots, T-1$, such that

$$a_j(n) = \sum_{r=0}^{T-1} e^{-ijr2\pi/T} R_x(n+r, r) = \int_0^{2\pi} e^{-inu} \gamma_j(du).$$

Proof: Define $\gamma_j(du) = (E(du)x, V^j x)$ $\square$

Vector measure $\gamma(\Delta) = [\gamma_0(\Delta), \gamma_1(\Delta), \ldots, \gamma_{T-1}(\Delta)] \in \mathcal{C}^T$ is called the *spectral measure* of $x(n)$. If $d\gamma = g(u)du$, then

$$g(u) = [g_0(u), g_1(u), \ldots, g_{T-1}(u)]$$

is called the *density* of $(x(n))$

$$a_j(n) = \int_0^{2\pi} e^{-inu} g_j(u) du, \quad j = 0, \ldots, T-1$$

Univariate Stationary Sequences

A T-PC sequence $(x(n))$ with $T = 1$ is called *stationary* ($V = I$, $x(n) = U^n x(0)$)

$$R_x(m, n) = R_x(m + r, n + r) = R_x(m - n, 0)$$

$$R_x(n, 0) = K_x(n) = \int_0^{2\pi} e^{-inu} F_x(du)$$

where F_x is the *spectral measure* of $(x(n))$. If $(x(n))$ is a.c, $F_x << dt$ then

$$K_x(n) = \int_0^{2\pi} e^{-inu} F(u) du$$

Any function h such that $F(u) = h(u)\overline{h(u)} = |h(u)|^2$ is called a *transfer function (t.f.)* of $(x(n))$. Then [Kolmogorov]

$$(x(n)) \approx (e^{-in\cdot} h(\cdot)) \in L^2(\mathcal{C})$$

Predicton Problem

Find t.f. h such that

1. analytic: $f(u) = \sum_{k=0}^{\infty} c_k e^{iku}$
2. outer (maximal):
   $\overline{span}\{e^{in\cdot} f(u) : n \geq 0\} = \overline{span}\{e^{in\cdot} : n \geq 0\} = L_+(\mathcal{C})$

The last condition means

$$\overline{span}\{x(n) : n \leq m\} = \overline{span}\{\xi_n : n \leq m\}$$

where (ξ_n) is an innovation. Consequently, if $M_x(m) = \overline{span}\{x(n) : n \leq m\}$, then the projection

$$P_{M_x(m)}(x(0)) = \sqrt{2\pi} \sum_{k=m}^{\infty} c_k \xi_{(-k)}$$

It is possible to express ξ_k in terms of $(x(n))$.

T-variate Stationary Sequences

T-variate stationary: $\mathbf{x}(n) = [x^k(n)] = \begin{bmatrix} x^0(n) \\ \vdots \\ x^{T-1}(n) \end{bmatrix}$, $x^k(n) \in \mathcal{H}$

Auto-covariance: $K_{\mathbf{x}}(n) = [(x^j(n), x^k(0))] = \mathbf{x}(n)\mathbf{x}(0)^*$
Spectral measure: $K_{\mathbf{x}}(n) = \int_0^{2\pi} e^{-inu}\mathbf{F}(du)$, $\mathbf{F}$ is matrix measure
A.c. sequence: $K_{\mathbf{x}}(n) = \int_0^{2\pi} e^{-inu}F(u)(du)$, $F(\cdot)$ is nonnegative.

Any $T \times T$ matrix function $H(\cdot)$ such that

$$F(u) = H(u)H(u)^*, \quad \text{where} \quad H(u)^* = \overline{H(u)'}$$

is called a *transfer function (t.f.)* of $(\mathbf{x}(n))$.

Predicton Problem

If H is a transfer function of $(\mathbf{x}(n))$, then

$$(\mathbf{x}(n)) \approx (e^{-in\cdot}H(\cdot)) \in L^2(\mathcal{C}^T)$$

Interpretation $H(t) = [H^{k,\cdot}(t)] = \begin{bmatrix} H^{0,\cdot}(t) \\ \vdots \\ H^{T-1,\cdot}(t) \end{bmatrix}$

Prediction Problem. Find H that is

1. analytic $H(u) = \sum_{k=0}^{\infty} C_k e^{iku}$
2. outer $\overline{span}\{e^{in\cdot}H(u) : n \leq 0\} = \overline{span}\{e^{in\cdot}I : n \leq 0\}$

Prediction Problem can be explicitly solved if coordinates of $F(u)$ are rational functions.

Rozanov's Theorem

If $n(z), d(z)$ are polynomials, then $h(z) = \dfrac{n(z)}{d(z)}$, $z \in \mathcal{C}$, is called *rational*. The function $h(e^{iu})$ is then called *rational function of* $u \in [0, 2\pi)$

For example $cos(u) = \dfrac{z + 1/z}{2} = \dfrac{z^2 + z}{2z}$, if $z = e^{iu}$

Theorem (Rozanov)

Each a.e. non-negative rational $T \times T$ matrix function $F(u)$, $u \in [0, 2\pi)$, of rank r can be represented in the form $F(u) = H(e^{iu})H(e^{iu})^*$ a.e. where $H(z)$ is rational, analytic and the rank of $H(z)$ is r for all z inside the open unit circle $D_{<1} = \{|z| < 1\}$, i.e. $H(z)$ has no zeros or poles in $D_{<1}$

If $r = T$ then $H(e^{iu})$ is outer, and since $H(z)$ is rational, $H(u) = A(z)^{-1}B(z)$ where $A(z)$ and $B(z)$ are left co-prime matrix polynomials

VARMA Systems

Stationary sequences with rational densities are exactly a.c. stationary solutions to VARMA systems:

$$\sum_{j=0}^{L} A_j \mathbf{x}(n-j) = \sum_{j=0}^{R} B_j \xi_{n-j}, \quad n \in \mathcal{Z}, \tag{2}$$

A_j, B_j are complex $T \times T$ matrices, $A_0, A_L, B_0, B_R \neq 0$, $\xi_n = [\xi_n^k]$ is such that $(\xi_n^k, \xi_m^j) = 1$ if $j = k$ and $m = n$, and 0 otherwise. We substitute $\mathbf{x}(n) = e^{-in\cdot} H(\cdot)$ and $\xi_n = e^{-in\cdot} I$. Then

$$\sum_{j=0}^{L} A_j e^{-i(n-j)\cdot} H(\cdot) = \sum_{j=0}^{R} B_j e^{-i(n-j)\cdot} I$$

,

$$e^{-in\cdot} \left(\sum_{j=0}^{L} A_j e^{ij\cdot} \right) H(\cdot) = e^{-in\cdot} \left(\sum_{j=0}^{R} B_j e^{ij\cdot} \right) I$$

Stationary Solution of VARMA System

Denote $\quad A(z) = \sum_{k=0}^{L} A(k)z^k, \quad B(z) = \sum_{k=0}^{R} B(k)z^k$

A t.f. of an a.c. stationary solution to VARMA system (2) is

$$H(u) = A(e^{iu})^{-1}B(e^{iu})$$

(if $A(e^{iu})^{-1}$ exists a.e.). Note that $H(u)$ is rational, so $F(u) = H(u)H(u)^*$ is rational.

OPPOSITE: Rozanov's theorem $\Rightarrow$ *if rank of $F(u)$, $r = T$, one can find rational, analytic and outer t.f. $H_0(u)$ and hence polynomial matrices $(A_0(z), B_0(z))$ with no zeros in $D_{<1}$ such that $F(u) = H_0(u)H_0(u)^*$ and $H_0(u) = A_0(e^{iu})^{-1}B_0(e^{iu})$*

$(A_0(z), B_0(z))$ is called a *VARMA representation (model)* for $(\mathbf{x}(n))$.

Transfer Function of a PC Sequence

Theorem (Makagon, Miamee 2013)

Let γ be the spectrum of an a.c T-PC sequence $(x(n))$. Then there exist a function $h \in L^2(\mathcal{C}^T)$ such that

$$g_j(u) = h(u)h(u + 2\pi j/T)^*$$

for every $j = 0, \ldots, T-1$.
The function h has the property that

$$x(n) \approx f(n)(u) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} e^{-inu} h(u + 2\pi j/T)$$

Compare with $x(n) = \frac{1}{T}\sum_{j=0}^{T-1} e^{-2\pi ijn/T} U^n V^j x.$

Corresponding T-variate Stationary Sequence

Let x(n) be T-PC

$$\ldots x(-1), \underbrace{x(0), x(1), \ldots, x(T-1)}_{\mathbf{X}(0)}, \underbrace{x(T), \ldots, x(2T-1)}_{\mathbf{X}(1)}, x(2T), \ldots$$

The T-variate stationary sequence

$$\mathbf{x}(n) = [x(nT), x(nT+1), \ldots, x((n+1)T-1)]', \quad n \in Z$$

is called the *T-variate stationary sequence corresponding* to $x(n)$.

Relations

Theorem (Makagon 2017)

Let h be a t.f. of $(x(n))$ and H be a t.f. of $(\mathbf{x}(n))$.

1. $h(u) = \sum_{k=0}^{T-1} e^{iku} H^{k\cdot}(Tu).$
2. *Given h define*

$$f_k(t) = (1/T) \sum_{j=0}^{T-1} e^{-ik(t+2\pi j/T)} h(t+2\pi j/T), \;\; k = 0, \ldots, T-1.$$

f_k *is* $2\pi/T$*-periodic,* $f_k(t) = h_k(Tt)$*. Then*

$$H^{k\cdot} = h_k$$

Relations for densities are also available [Makagon 2017].

PARMA System

A PARMA system is a system of difference equations

$$x(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \qquad n \in \mathcal{Z}, \qquad (3)$$

where $a_j(n), b_j(n) \in \mathcal{C}$ are T-periodic in n, none of the sequences $(a_l(n))$, and $(b_r(n))$ are identically zero, and (ξ_n) are orthonormal. The system above can be written as

$$\sum_{j=0}^{l} a_j(n)x(n-j) = \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \quad n \in \mathcal{Z}.$$

We arrange $a_j(t)$ in a matrix $[A_L \ \ldots \ A_1 \ A_0]$ as follows

$$\left[\begin{array}{c|ccc|cccc} \ldots & a_T(0) & \ldots & a_1(0) & a_0(0) & 0 & \ldots & 0 \\ \ldots & a_{T+1}(1) & \ldots & a_2(1) & a_1(1) & a_0(1) & \ldots & 0 \\ \ldots & \ldots & \mathbf{A(1)} & \ldots & \ldots & \mathbf{A(0)} & \ldots & \ldots \\ \ldots & \ldots & \ldots & a_T(T-1) & a_{T-1}(T-1) & \ldots & \ldots & a_0(T-1) \end{array}\right],$$

and do the same for $b_j(t)$'s defining $[B_R \ \ldots \ B_1 \ B_0]$.

Then we can write a PARMA system (3)as VARMA on $(\mathbf{x}(n))$

$$\sum_{j=0}^{L} A_j \mathbf{x}(n-j) = \sum_{j=0}^{R} B_j \xi_{n-j}, \quad n \in \mathcal{Z},$$

where $(\mathbf{x}(n))$ and (ξ_n) are T-variate stationary corresponding to $(x(n))$ and (ξ_n).

Recap

Given an a.c. T-PC sequence $(x(n))$ with density $g(t)$. We know:

1. how to connect $(x(n))$ with the corresponding T-variate stationary sequence $(\mathbf{x}(n))$
2. how to connect the parameters of $(x(n))$ and $(\mathbf{x}(n))$, in particular how to find the density $F(t)$ of $(\mathbf{x}(n))$
3. how to find an analytic t.f. $H(t)$ of $(\mathbf{x}(n))$, if $F(t)$ is rational
4. $F(t)$ is rational iff $g(t)$ is rational
5. how to connect a PARMA system on $(x(n))$ with a VARMA system on $(\mathbf{x}(n))$
6. how to solve a VARMA system on $(\mathbf{x}(n))$

Everything is ready study T-PC squences with rational density.

An Example of Theorem

Theorem (Makagon, 2017)

Suppose that $(x(n))$ jest T-PC with rational density. Then there is a PARMA system $(A(z), B(z))$ such that

1. *all zeros of $A(z)$ are outside the disk $D_{\leq 1} = \{z \in \mathcal{C} : |z| \leq 1\}$*
2. *all zeros of of $B(z)$ are outside the open disk $D_{<1} = \{z \in \mathcal{C} : |z| < 1\}$*
3. *polynomial matrices $A(z)$ and $B(z)$ are left co-prime*
4. *$(x(n))$ is the only T-PC solution to the system $(A(z), B(z))$*

The system $(A(z), B(z))$ above is a PARMA representation (model) of $(x(n))$.

Procedure 1: Solving PARMA System

Given PARMA system

$$x(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \qquad n \in \mathcal{Z}, \qquad (4)$$

- compute $A(z)$ and $B(z)$, (as above)
- if $\det(A(z) = 0$ for some z of modulus one, then the system has no unique a.c. T-PC solution;
- otherwise we compute $H(t) = A(e^{it})^{-1}B(e^{it})$;
- compute $h(t) = \sum_{k=0}^{T-1} e^{ikt} H^{k\cdot}(Tt)$;
- compute $g^j(t) = h(t)h(t+2\pi j/T)^*$, $j = 0, \ldots, T-1$.

The function $g(t) = (g^0(t), ..., g^{T-1}(t))$ is the density of an a.c T-PC solution to the PARMA system (4)

Procedure 2: Finding PARMA Representation

Given rational density g of a T-PC $(x(n))$

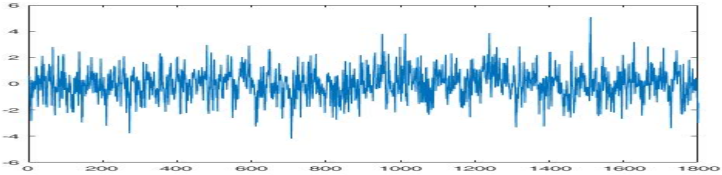
- compute the density $F(t)$ of $(\mathbf{x}(n))$ [Makagon 2017]
- use Rozanov to find an analytic t.f. $H(t)$
- represent $H(t)$ as the quotient $H(t) = A(e^{it})^{-1}B(e^{it})$, where $A(z)$ and $B(z)$ are left co-prime and have no zeros in the unit circle $D_{<1}$
- adjust $A(z)$ and $B(z)$ so that $A(0)$ and $B(0)$ are left diagonal and $A(0)$ has ones on the diagonal

Adjusted $((A(z), B(z))$ is a *PARMA representation* of $(x(n))$

REMARK. Impossible to effectively compute.Several tries were carried out in analysis of MIMO models in signal processing.

Example

Assume that our data x_n, $n = 0, \ldots, 1799$, came from a T-PC $(x(n))$ with $T = 3$



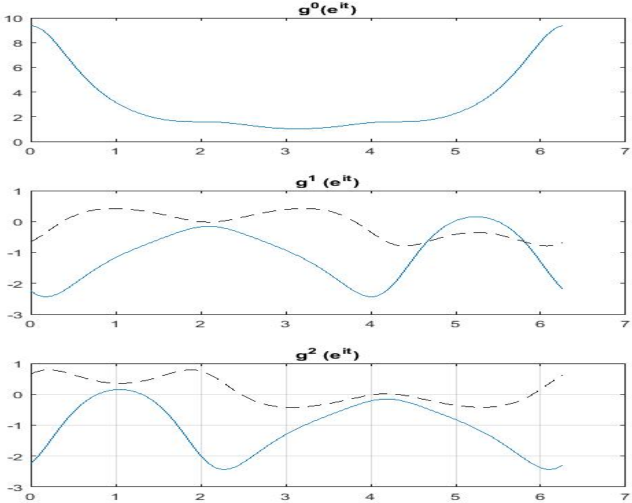
We guess [?] that a model for this data is

$$
\begin{aligned}
x(0) &= 0.13x(-3) + 0.3\xi_{-1} + 0.5\xi_0 \\
x(1) &= -0.1x(-2) + 0.26\xi_{-2} + 0.42\xi_{-1} + 0.95\xi_0 + 0.50\xi_1, \\
x(2) &= -0.32x(-1) + 0.40\xi_{-1} + 0.21\xi_0 + 0.55\xi_1 + 0.9924\xi_2
\end{aligned}
$$

Is this model plausible?

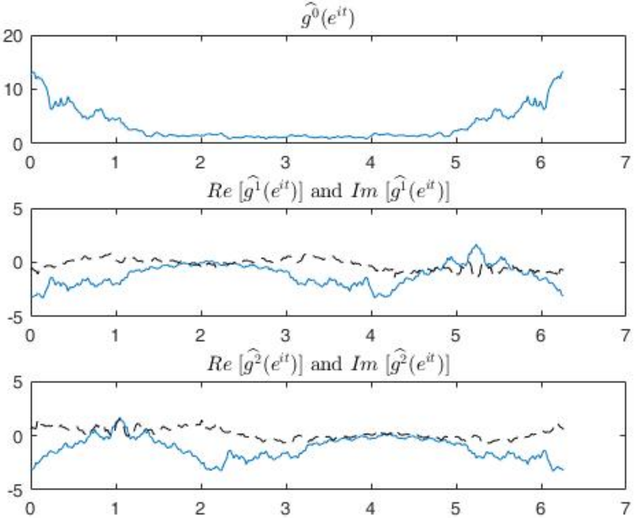
Example, cont.

We compute the solution to the above system (Procedure 1)



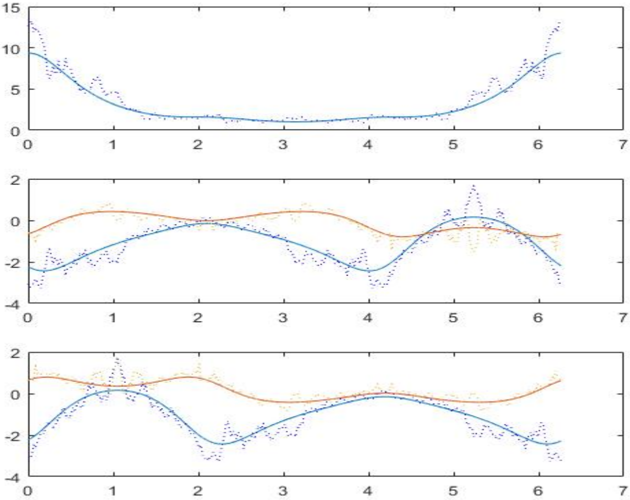
Example. cont.

We compute the Hurd's periodogram [Hurd, Miamee]



Example, cont.

We graph them together



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THANK YOU