Irregular Almost-Cyclostationary Model for the Sunspot Number Time Series

Antonio Napolitano

Department of Engineering
University of Napoli "Parthenope"
Napoli, Italy

Email: antonio.napolitano@uniparthenope.it antnapol@gmail.com
URL: https://sites.google.com/site/antnapol

work in collaboration with William A. Gardner, University of California, Davis, CA, USA

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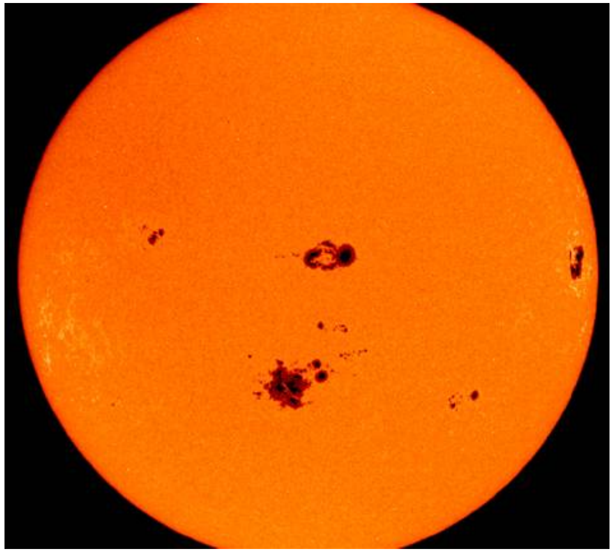
Abstract

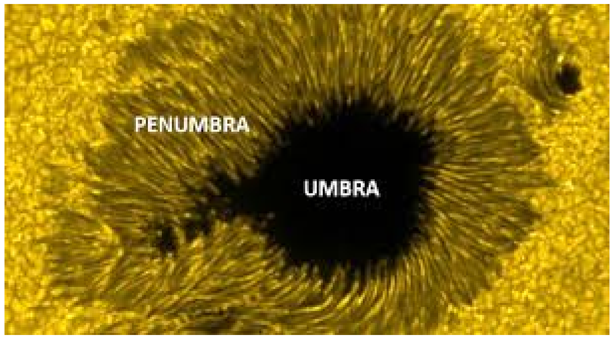
The time series of Sunspot number is known to exhibit approximate periodicity. In the brief study of this time series provided here, the details of the irregularity in the periodicity are exposed by fitting an irregular almost-cyclostationary model to the data. By two different experiments, it is shown that from the second-order lag product of the Sunspot number time series, two amplitude- and angle-modulated additive sine-wave components can be extracted. The periods of the non-modulated sinusoids agree with those already observed. Moreover, the time-warping functions in the model provide a mathematical description of the irregularity of the cyclicities observed in the time series, something not previously attempted.

Outline

- Sunspots
- Sunspot Number Time Series
- Oscillatory almost-cyclostationary processes
- Amplitude-modulated and time-warped almost-cyclostationary signals
- Amplitude-modulating and time-warping function measurements
- Amplitude-modulation compensation and de-warping
- 27.3-day irregular period
- 11-year irregular period
- 120-200-year irregular period
- Conclusion

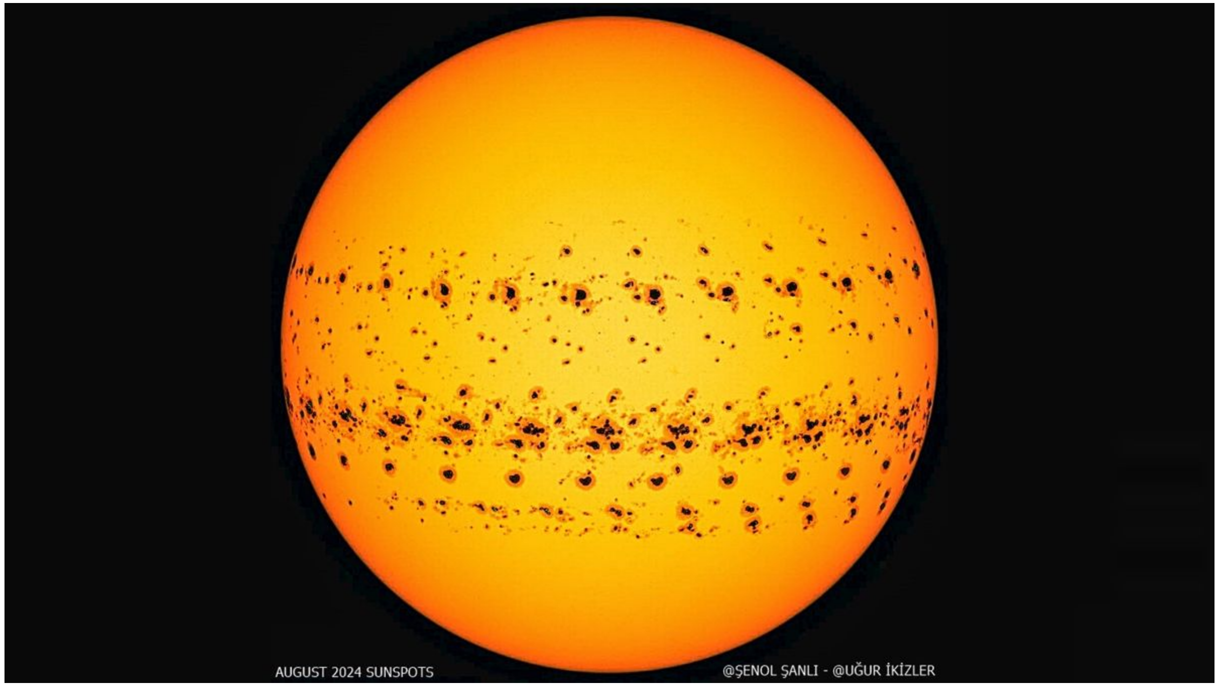
Sunspots





- Sunspots are temporary spots on the Sun's surface that are darker (and cooler) than the surrounding area. They typically last for several days, although very large ones may live for several weeks.
- Sunspots are regions with magnetic fields whose strengths are thousands of times stronger than the Earth's magnetic field.
- Sunspots usually come in groups with two sets of spots. One set has positive or north magnetic field while the other set has negative or south magnetic field.
- The number of Sunspots characterizes the solar activity that disturbs radio communications, the orbits of satellites, and power grids.

Sunspots (cont'd)



During August 2024, there was an average of 215.5 daily sunspots on the Sun's surface. This **time-lapse image** shows every visible dark patch moving across the sun during this time. (Image credit: SDO/Senol Sanli/Ugur Ikizler)

Sunspot Number Time Series

The relative Sunspot number R (also known as Wolf number or Zürich number) is a quantity that measures the number of sunspots and groups of Sunspots present daily or monthly on the surface of the Sun.

$$R = k\,(10g + s)$$

where

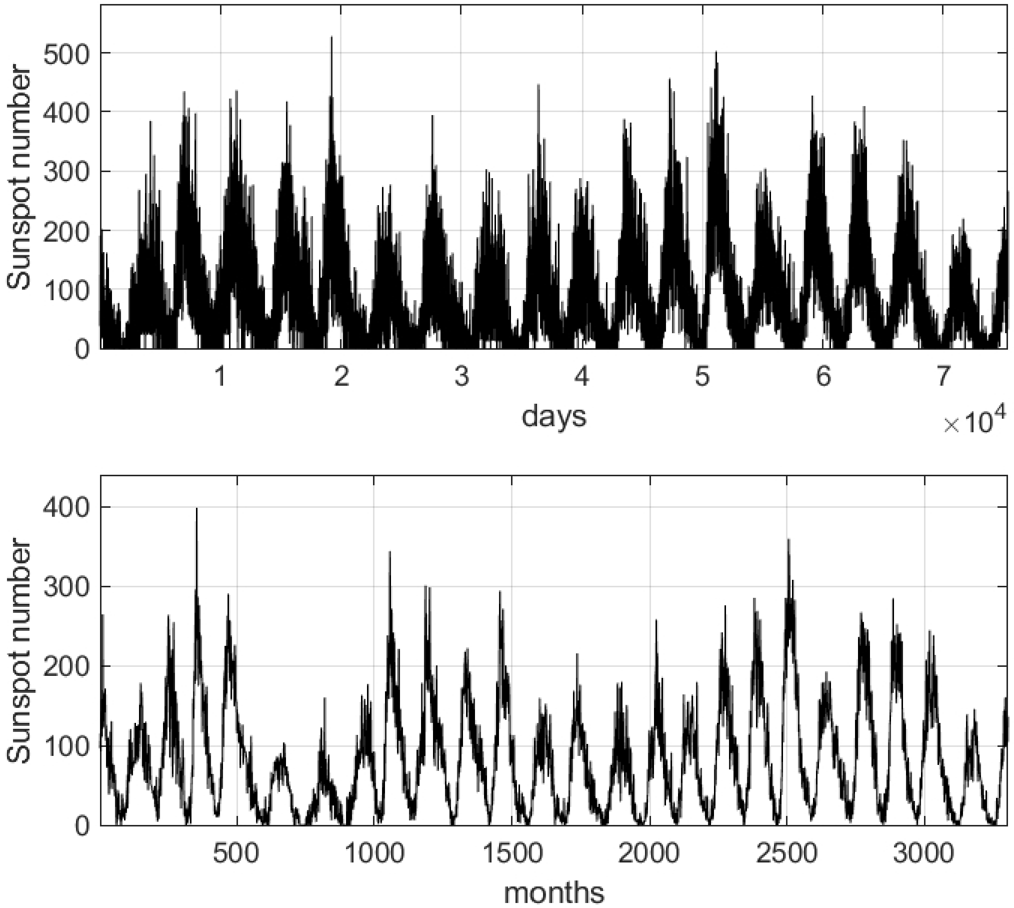
- s = number of individual spots
- g = number of sunspot groups
- k = factor that varies with observer and is referred to as the observatory factor or the personal reduction coefficient

13-month smoothed monthly mean sunspot number

$$\overline{R}_m = (0.5R_{m-6} + R_{m-5} + \cdots + R_{m-1} + R_m + R_{m+1} + \cdots + R_{m+5} + 0.5R_{m+6})/12$$

R_{m+n} = monthly sunspot number n months away from month m

Sunspot Number Time Series (cont'd)



(Top) daily total Sunspot number R in the years 1818–2023.
(Bottom) monthly mean total Sunspot number $\overline{R}_m$ in the years 1749–2023.

Motivation

- Combining a periodic phenomenon and a random one (in a way that does not modify the time scale of the periodic phenomenon) leads to a process which is not periodic, but whose probabilistic functions (distribution, mean, characteristic function, autocorrelation, moments, cumulants) are periodic functions of time. Such a process is called ***cyclostationary (CS)***.
- If more periodicities with incommensurate periods are present, then the statistical characteristics are almost-periodic functions of time and the process is called ***almost-cyclostationary (ACS)***. The periodic or almost periodic probabilistic functions are expressed as sum of complex sinewaves whose amplitudes and phases depend on the lag parameters.
- Applications in telecommunications, telemetry, radar and sonar, mechanics, radio astronomy, Earth sciences, econometrics.
- Many physical phenomena give rise to signals with statistical functions exhibiting ***irregular cyclicity***: intentional or unintentional **time- or frequency-warping** due to variations ot timing parameters (carrier frequency, baud rate); **Doppler effect** due to relative motion between transmitter and receiver with generic motion law; irregular pace in **biological signals** such as electrocardiogram (ECG) or electroencephalogram (EEG).

Almost-Cyclostationary Processes

Statistical functions (e.g., distribution function, mean value, characteristic function, auto-correlation function, autocovariance function) are **almost-periodic functions** of time.

$$F_x(t;\xi) \triangleq \mathrm{E}\{\mathrm{u}(\xi - x(t))\} = \sum_{\gamma \in \Gamma^{(1)}} F_x^{\gamma}(\xi)\, e^{j2\pi\gamma t}$$

$$m_x(t) \triangleq \mathrm{E}\{x(t)\} = \sum_{\eta \in \mathscr{E}^{(1)}} x_\eta\, e^{j2\pi\eta t}$$

$$\Phi_x(t;\omega) \triangleq \mathrm{E}\left\{e^{j\omega x(t)}\right\} = \sum_{\gamma \in \Gamma^{(1)}} \Phi_x^{\gamma}(\omega)\, e^{j2\pi\gamma t}$$

$$R_{\boldsymbol{x}}(t;\tau) \triangleq \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\} = \sum_{\alpha \in \mathscr{A}} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{j2\pi\alpha t}$$

$$C_{\boldsymbol{x}}(t;\tau) \triangleq \mathrm{E}\left\{[x(t+\tau) - m_x(t+\tau)]\, [x(t) - m_x(t)]^{(*)}\right\} = \sum_{\beta \in \mathscr{B}} C_{\boldsymbol{x}}^{\beta}(\tau)\, e^{j2\pi\beta t}$$

Almost-Cyclostationary Processes (cont'd)

Cyclic CDF

$$F_x^{\gamma}(\xi) \triangleq \left\langle \mathrm{E}\{\mathrm{u}(\xi - x(t))\}\, e^{-j2\pi\gamma t} \right\rangle_t$$

Cyclic Mean

$$x_{\eta} \triangleq \left\langle \mathrm{E}\left\{x(t)\right\}\, e^{-j2\pi\eta t} \right\rangle_t$$

Cyclic Characteristic Function

$$\Phi_x^{\gamma}(\omega) = \left\langle \Phi_x(t;\omega)\, e^{-j2\pi\gamma t} \right\rangle_t$$

(Conjugate) Cyclic Autocorrelation Function

$$R_{\boldsymbol{x}}^{\alpha}(\tau) \triangleq \left\langle \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\} e^{-j2\pi\alpha t} \right\rangle_t$$

(Conjugate) Cyclic Autocovariance Function

$$C_{\boldsymbol{x}}^{\beta}(\tau) \triangleq \left\langle \mathrm{E}\left\{ [x(t+\tau) - \mathrm{E}\{x(t+\tau)\}]\, [x(t) - \mathrm{E}\{x(t)\}]^{(*)} \right\} e^{-j2\pi\beta t} \right\rangle_t$$

Almost-Cyclostationary Processes (cont'd)

Cramér Spectral Representation

$$x(t) = \int_{\mathbb{R}} e^{j2\pi f t} \, \mathrm{d}Z_x(f)$$

$Z_x(f)$ ***integrated complex spectrum: process with increments correlated if*** $f_1 + {\scriptstyle(-)} f_2 = \alpha \in \mathscr{A}$

$$\mathrm{E}\left\{ \mathrm{d}Z_x(f_1) \, \mathrm{d}Z_x^{(*)}(f_2) \right\} = \sum_{\alpha \in \mathscr{A}} \delta(f_2 + {\scriptstyle(-)}(f_1 - \alpha)) \, \mathrm{d}\mu_{\boldsymbol{x}}^{\alpha}(f_1) \, \mathrm{d}f_2$$

If the complex measure $\mu_{\boldsymbol{x}}^{\alpha}(f)$ does not contain a singular component,

$$\mathrm{d}\mu_{\boldsymbol{x}}^{\alpha}(f) = S_{\boldsymbol{x}}^{\alpha}(f) \, \mathrm{d}f$$

with $S_{\boldsymbol{x}}^{\alpha}(f)$ (conjugate) cyclic spectrum of $x(t)$ possibly containing Dirac deltas in correspondence of the jumps in $\mu_{\boldsymbol{x}}^{\alpha}(f)$.

$$S_{\boldsymbol{x}}^{\alpha}(f) = \mathscr{F}\left[R_{\boldsymbol{x}}^{\alpha}(\tau)\right]$$

${\scriptstyle(-)}$ = optional minus sign linked to the optional complex conjugation $(*)$

Oscillatory Almost-Cyclostationary (OACS) Processes

Statistical functions (e.g., distribution function, mean value, characteristic function, autocorrelation, autocovariance) are the **superposition of amplitude- and angle-modulated sine waves**.

$$F_y(t;\xi) \triangleq \mathrm{E}\{\mathrm{u}(\xi - y(t))\} = \sum_{\gamma\in\Gamma^{(1)}} F_y^{\gamma}(t;\xi)\, e^{j2\pi\gamma t}$$

$$m_y(t) \triangleq \mathrm{E}\{y(t)\} = \sum_{\eta\in\mathscr{E}^{(1)}} m_y^{\eta}(t)\, e^{j2\pi\eta t}$$

$$\Phi_y(t;\omega) \triangleq \mathrm{E}\left\{e^{j\omega y(t)}\right\} = \sum_{\gamma\in\Gamma^{(1)}} \Phi_y^{\gamma}(t;\omega)\, e^{j2\pi\gamma t}$$

$$R_{\boldsymbol{y}}(t;\tau) \triangleq \mathrm{E}\{y(t+\tau)\, y^{(*)}(t)\} = \sum_{\alpha\in\mathscr{A}} \rho_{\boldsymbol{y}}^{\alpha}(t;\tau)\, e^{j2\pi\alpha t}$$

$$C_{\boldsymbol{y}}(t;\tau) \triangleq \mathrm{E}\{[y(t+\tau) - m_y(t+\tau)]\, [y(t) - m_y(t)]^{(*)}\} = \sum_{\beta\in\mathscr{B}} \kappa_{\boldsymbol{y}}^{\beta}(t;\tau)\, e^{j2\pi\beta t}$$

Oscillatory Almost-Cyclostationary (OACS) Processes (cont'd)

The magnitude an phase of the

- ***Evolutionary Cyclic CDF*** $F_y^{\gamma}(t;\xi)$
- ***Evolutionary Cyclic Mean*** $m_y^{\eta}(t)$
- ***Evolutionary Cyclic Characteristic Function*** $\Phi_y^{\gamma}(t;\omega)$
- ***Evolutionary (Conjugate) Cyclic Autocorrelation Function*** $\rho_{\boldsymbol{y}}^{\alpha}(t,\tau)$
- ***Evolutionary (Conjugate) Cyclic Autocovariance Function*** $\kappa_{\boldsymbol{y}}^{\beta}(t,\tau)$

are the amplitude- and phase-modulation functions of the modulated sine wave at frequency γ, η, γ, α, β, respectively, of the CDF, mean, chacarteristic function, (conjugate) autocorrelation, and (conjugate) autocovariance, respectively.

Oscillatory Almost-Cyclostationary (OACS) Processes (cont'd)

Priestley Spectral Representation

$$y(t) = \int_{\mathbb{R}} A_t(f)\, e^{j2\pi f t}\, \mathrm{d}Z_x(f)$$

$Z_x(f)$ ***integrated complex spectrum of an ACS process*** $x(t)$

$$\mathrm{E}\left\{\mathrm{d}Z_x(f_1)\, \mathrm{d}Z_x^{(*)}(f_2)\right\} = \sum_{\alpha \in \mathscr{A}} \delta(f_2 +_{(-)} (f_1 - \alpha))\, \mathrm{d}\mu_{\boldsymbol{x}}^{\alpha}(f_1)\, \mathrm{d}f_2$$

$\{\mu_{\boldsymbol{x}}^{\alpha}(f_1)\}_{\alpha \in \mathscr{A}}$ = family of complex spectral measures

$A_t(f)$ ***deterministic modulating function*** – low-pass function (as function of t)

non univocal representation: $A_t(f)\, \mathrm{d}Z_x(f) = [A_t(f)/k]\, \mathrm{d}[k\, Z_x(f)]$

Oscillatory Almost-Cyclostationary (OACS) Processes (cont'd)

(Conjugate) Autocorrelation Function

$$\boxed{\mathrm{E}\left\{y(t+\tau)\,y^{(*)}(t)\right\} = \sum_{\alpha\in A} \rho_{\boldsymbol{y}}^{\alpha}(t,\tau)\, e^{j2\pi\alpha t} = \sum_{\alpha\in A} \left|\rho_{\boldsymbol{y}}^{\alpha}(t,\tau)\right|\, e^{j[2\pi\alpha t+\arg\{\rho_{\boldsymbol{y}}^{\alpha}(t,\tau)\}]}}$$

Evolutionary (Conjugate) Cyclic Autocorrelation Function

$$\rho_{\boldsymbol{y}}^{\alpha}(t,\tau) = \int_{\mathbb{R}} A_{t+\tau}(f)\, A_t^{(*)}((-)(\alpha-f))\, e^{j2\pi f\tau}\, \mathrm{d}\mu_{\boldsymbol{x}}^{\alpha}(f)$$

The (conjugate) autocorrelation is the superposition of amplitude- and angle-modulated complex sinewaves whose amplitude- and angle-modulating functions are the magnitudes and phases of the evolutionary (conjugate) cyclic autocorrelation functions.

Evolutionary (Conjugate) Cyclic Spectrum

$$\mathrm{d}\sigma_t^{\alpha}(f) = A_t(f)\, A_t^{(*)}((-)(\alpha-f))\, \mathrm{d}\mu_{\boldsymbol{x}}^{\alpha}(f)$$

Amplitude-Modulated Time-Warped (AMTW) Almost-Cyclostationary (ACS) Processes

$$\boxed{y(t) = a(t)\, x(t + \varepsilon(t))}$$

- $x(t)$ = (underlying) almost-cyclostationary process
- $a(t)$ = amplitude-modulation function
- $\varepsilon(t)$ = time-warping function
- Special case of OACS processes
- $A_t(f) = a(t)\, e^{j2\pi f \varepsilon(t)}$

Amplitude-Modulated Time-Warped (AMTW) Almost-Cyclostationary (ACS) Processes (cont'd)

Statistical functions (e.g., distribution function, mean value, characteristic function, autocorrelation function) are the **superposition of amplitude- and angle-modulated sine waves**.

$$F_y(t;\xi) \triangleq \mathrm{E}\{\mathrm{u}(\xi - y(t))\} = \sum_{\gamma \in \Gamma^{(1)}} F_x^{\gamma}(\xi / a(t))\, e^{j2\pi\gamma(t+\varepsilon(t))}$$

$$m_y(t) \triangleq \mathrm{E}\{y(t)\} = \sum_{\eta \in \mathscr{E}^{(1)}} a(t)\, x_\eta\, e^{j2\pi\eta(t+\varepsilon(t))}$$

$$\Phi_y(t;\omega) \triangleq \mathrm{E}\left\{e^{j\omega y(t)}\right\} = \sum_{\gamma \in \Gamma^{(1)}} \Phi_x^{\gamma}(\omega\, a(t))\, e^{j2\pi\gamma(t+\varepsilon(t))}$$

$$R_{\boldsymbol{y}}(t;\tau) \triangleq \mathrm{E}\{y(t+\tau)\, y^{(*)}(t)\} = \sum_{\alpha \in \mathscr{A}} a(t+\tau)\, a^{(*)}(t)\, R_{\boldsymbol{x}}^{\alpha}\big(\tau + \varepsilon(t+\tau) - \varepsilon(t)\big)\, e^{j2\pi\alpha(t+\varepsilon(t))}$$

Amplitude-Modulated Time-Warped (AMTW) Almost-Cyclostationary (ACS) Processes (cont'd)

$\varepsilon(t)$ slowly varying with respect to t (whose derivative is 1)

$$\sup_t \left| \dot{\varepsilon}(t) \right| \ll 1$$

$a(t)$ real valued, positive
$a(t+\tau) \simeq a(t)$ for all τ such that $R_{\boldsymbol{x}}^{\alpha}(\tau)$ is significantly non zero

(Conjugate) Autocorrelation Function

$$\boxed{\mathrm{E}\left\{ y(t+\tau)\, y^{(*)}(t) \right\} \simeq a^2(t) \sum_{\alpha \in \mathscr{A}} e^{j2\pi\alpha\varepsilon(t)} R_{\boldsymbol{x}}^{\alpha}(\tau)\, e^{j2\pi\alpha t}}$$

Statistical Function Estimation

- Statistical functions of the amplitude-modulated and time-warped signal $y(t)$ are not directly estimated. Rather, their components are estimated.
- The amplitude modulation function $a(t)$ and time-warping function $\varepsilon(t)$ are estimated;
- An estimate $\widehat{x}(t)$ of the underlying ACS signal $x(t)$ is obtained by compensating $a(t)$ and $\varepsilon(t)$ (de-warping) in the available data $y(t)$;
- Statistical functions (cyclic autocorrelation, cyclic spectrum, cyclic CDF, cyclic characteristic function) of $x(t)$ are estimated starting from $\widehat{x}(t)$.

Amplitude-Modulation Function and Warping Function Estimation

$$\overline{Q}_y(t;\theta) = \mathrm{E}\{Q_y(t;\theta)\} = \sum_{\gamma\in\Gamma} q_x^\gamma(t;\theta)\, e^{j2\pi\gamma(t+\varepsilon(t))}$$

	$\overline{Q}_y(t;\theta)$	$Q_y(t;\theta)$	$q_x^\gamma(t;\theta)$	θ	Γ
CDF	$F_y(t;\xi)$	$\mathrm{u}(\xi - y(t))$	$F_x^\gamma(\xi/a(t))$	ξ	$\Gamma^{(1)}$
mean	$m_{y,1}(t)$	$y(t)$	$a(t)\, x_\eta$		$\mathscr{E}^{(1)}$
characteristic function	$\Phi_y(t;\omega)$	$e^{j\omega y(t)}$	$\Phi_x^\gamma(\omega\, a(t))$	ω	$\Gamma^{(1)}$
autocorrelation	$R_{\boldsymbol{y}}(t;\tau)$	$y(t+\tau)\, y^{(*)}(t)$	$a^2(t)\, R_{\boldsymbol{x}}^\alpha(\tau)$	τ	$\mathscr{A}$

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

Assumption: There exist a cycle frequency $\gamma_0 \in \Gamma$ and a (fixed) value of θ such that the power spectrum of the modulated sine wave $q_x^{\gamma_0}(t;\theta)\, e^{j2\pi\gamma_0(t+\varepsilon(t))}$ has approximated bandwidth W and does not significantly overlap the power spectra of the other modulated sine waves at frequencies $\gamma \neq \gamma_0$.

$$\boxed{\begin{aligned} z^{(\gamma_0,W)}(t;\theta) &\triangleq \left[Q_y(t;\theta)\, e^{-j2\pi\gamma_0 t}\right] \otimes h_W(t) \\ &\simeq q_x^{\gamma_0}(t;\theta)\, e^{j2\pi\gamma_0\varepsilon(t)} \end{aligned}}$$

$h_W(t)$ = low-pass filter with bandwidth W

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

Assumption: $\arg[q_x^{\gamma_0}(t;\theta)]$ varies slowly with respect to $2\pi\gamma_0\varepsilon(t)$:

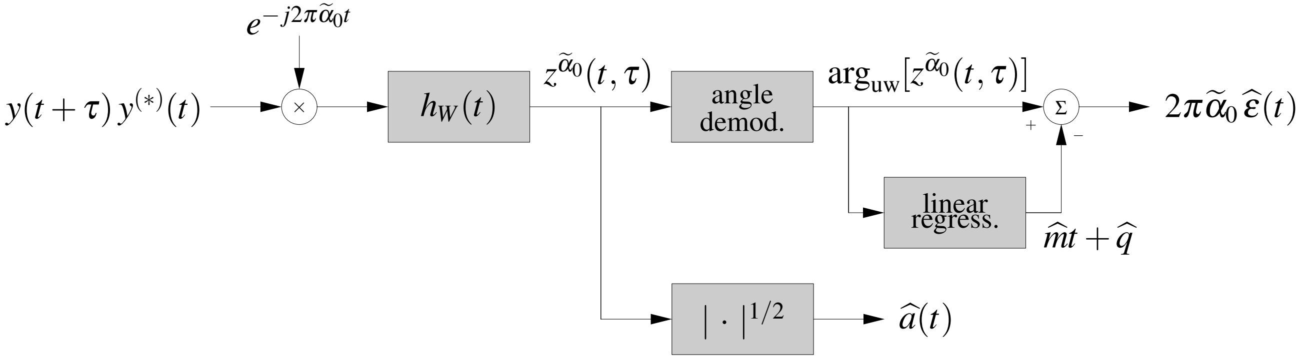
$$\sup_t \left| \frac{\partial}{\partial t} \arg[q_x^{\gamma_0}(t;\theta)] \right| \ll 2\pi|\gamma_0| \sup_t |\dot{\varepsilon}(t)|$$

$$\boxed{\widehat{\varepsilon}(t) = \frac{1}{2\pi\gamma_0} \arg_{\mathrm{uw}} \left[z^{(\gamma_0,W)}(t;\theta) \right]}$$

- $\arg_{\mathrm{uw}}[\cdot]$ = unwrapped phase
- If γ_0 is unknown, it is replaced by an estimate.
- The estimate of $\varepsilon(t)$ can be refined by a two step procedure.

$$\boxed{\widehat{a}(t) = \begin{cases} |z^{(\gamma_0,W)}(t;\theta)|^{1/2} & \text{if } \overline{Q}_y(t;\theta) = \text{autocorrelation} \\ |z^{(\gamma_0,W)}(t;\theta)| & \text{if } \overline{Q}_y(t;\theta) = \text{mean} \end{cases}}$$

Amplitude-Modulation Function and Warping Function Estimation (cont'd)



$$z^{(\widetilde{\alpha}_0,W)}(t,\tau) \triangleq \left[y(t+\tau)\,y^{(*)}(t)\,e^{-j2\pi\widetilde{\alpha}_0 t}\right] \otimes h_W(t)$$

$$\simeq a^2(t)\,R_{\boldsymbol{x}}^{\alpha_0}(\tau)\,e^{j2\pi\alpha_0\varepsilon(t)}\,e^{j2\pi(\alpha_0-\widetilde{\alpha}_0)t}$$

$$\widetilde{\alpha}_0 = \text{coarse estimate of } \alpha_0 \in A-\{0\}$$

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

Assumptions
$$\begin{cases} \widetilde{\alpha}_0 = \text{coarse estimate of } \alpha_0 \text{ assumed unknown} \\ a(t) = \text{real and positive} \\ |\alpha_0 - \widetilde{\alpha}_0| + B^{\alpha_0} < W < \inf\limits_{\substack{\alpha \in A \\ \alpha \neq \alpha_0}} \left(|\alpha - \alpha_0| - B^{\alpha}\right) - |\alpha_0 - \widetilde{\alpha}_0| \,. \end{cases}$$

B^{α} = monolateral bandwidth of $a^2(t)\, e^{j2\pi\alpha\varepsilon(t)}$

W = monolateral bandwidth of the low-pass filter $h_W(t)$

$\arg_{\mathrm{uw}}[\cdot]$ = unwrapped phase

$$\arg_{\mathrm{uw}}\left[\mathrm{E}\left\{z^{(\widetilde{\alpha}_0,W)}(t,\tau)\right\}\right] \simeq 2\pi\alpha_0\varepsilon(t) + \underbrace{2\pi(\alpha_0 - \widetilde{\alpha}_0)t + \arg\left[R_{\boldsymbol{x}}^{\alpha_0}(\tau)\right]}_{\text{affine term} \quad mt+q}$$

$\varepsilon(t)$ is estimated to within an unknown constant representing a fixed time delay

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

1) coarse estimate $\widetilde{\alpha}_0$ available (e.g., by locating the peaks of the PSD of $y(t+\tau)\, y^{(*)}(t)$)

2) $z^{(\widetilde{\alpha}_0,W)}(t,\tau) \triangleq \left[y(t+\tau)\, y^{(*)}(t)\, e^{-j2\pi\widetilde{\alpha}_0 t}\right] \otimes h_W(t)$

3) $\mathrm{E}\left\{z^{(\widetilde{\alpha}_0,W)}(t,\tau)\right\} \simeq a^2(t)\, R_{\boldsymbol{x}}^{\alpha_0}(\tau)\, e^{j2\pi\alpha_0\varepsilon(t)}\, e^{j2\pi(\alpha_0-\widetilde{\alpha}_0)t}$

provided that

$|\widetilde{\alpha}_0-\alpha_0|+B^{\alpha_0} < W+|\widetilde{\alpha}_0-\alpha_0| < \inf\limits_{\substack{\alpha\in A\\ \alpha\neq\alpha_0}} \left(|\alpha-\widetilde{\alpha}_0|-B^{\alpha}\right)$

4) $\arg\left[\mathrm{E}\left\{z^{(\widetilde{\alpha}_0,W)}(t,\tau)\right\}\right] \simeq \arg\left[R_{\boldsymbol{x}}^{\alpha_0}(\tau)\right]+2\pi\alpha_0\varepsilon(t)+2\pi(\alpha_0-\widetilde{\alpha}_0)t \mod 2\pi$

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

5) Estimate the affine term $mt+q$ in $\arg[\mathrm{E}\{z^{(\widetilde{\alpha}_0,W)}(t,\tau)\}]$. Estimates $\widehat{m}$ and $\widehat{q}$ obtained by least-squares linear regression on the available data $\arg_{\mathrm{uw}}[z^{(\widetilde{\alpha}_0,W)}(t,\tau)]$ ($t=nT_s$, $n=,0,1,\ldots,N-1$)

6) Estimate α_0 as $\widehat{\alpha}_0 = \dfrac{\widehat{m}}{2\pi} + \widetilde{\alpha}_0$

$$\boxed{\widehat{a}(t) = \left| z^{(\widetilde{\alpha}_0,W)}(t,\tau) \right|^{1/2}}$$

$$\boxed{\widehat{\varepsilon}(t) = \frac{1}{2\pi\widehat{\alpha}_0}\left[\arg_{\mathrm{uw}}\left[z^{(\widetilde{\alpha}_0,W)}(t,\tau)\right] - (\widehat{m}t+\widehat{q})\right]}$$

Amplitude-Modulation Function and Warping Function Estimation (cont'd)

- $\arg_{\mathrm{uw}}[\cdot]$ = unwrapped phase
- $\varepsilon(t)$ is estimated to within the unknown constant $\arg[R_{\boldsymbol{x}}^{\alpha_0}(\tau)]/(2\pi\alpha_0)$ = constant delay
- $a(t)$ is estimated but for a constant (with respect to t) multiplicative factor $|R_{\boldsymbol{x}}^{\alpha_0}(\tau)|$
- Warping-function estimation without an iterative search
- The estimation procedure can be repeated by replacing $\widetilde{\alpha}_0$ with $\widehat{\alpha}_0$ (obtained at item 6))

Amplitude-Modulation Compensation and De-Warping

$$\psi(t) = t + \varepsilon(t) \text{ with } \varepsilon(t) \text{ slowly varying } (\sup_t |\dot{\varepsilon}(t)| \ll 1)$$

$$\Downarrow$$

$$\psi^{-1}(t) \text{ can be approximated as } \psi^{-1}(t) \simeq t - \varepsilon(t)$$

if estimates $\widehat{a}(t)$ and $\widehat{\varepsilon}(t)$ are available,

then an estimate of the underlying ACS process $x(t)$ is

$$\boxed{\widehat{x}(t) = \frac{y(\widehat{\psi}^{-1}(t))}{\widehat{a}(\widehat{\psi}^{-1}(t))} \simeq \frac{y(t-\widehat{\varepsilon}(t))}{\widehat{a}(t-\widehat{\varepsilon}(t))}}$$

for all t such that $\widehat{a}(t-\widehat{\varepsilon}(t)) \neq 0$,

provided that $\sup_t |\varepsilon(t) - \widehat{\varepsilon}(t)| \ll 1/B$, B = bandwidth of $x(t)$

<u>$x(t)$ is estimated but for a constant delay and a constant multiplicative factor</u>

Amplitude-Modulation Compensation and De-Warping (cont'd)

Interpolating Formula

$$y(nT_s - \widehat{\varepsilon}(nT_s)) = \sum_{k=-N/2}^{N/2} y(kT_s)\,\mathrm{sinc}\left(\frac{nT_s - \widehat{\varepsilon}(nT_s)}{T_s} - k\right)$$

Measurement of Statistical Functions of the Underlying ACS Process

(conjugate) cyclic correlogram of $\widehat{x}(t)$ =
estimate of the (conjugate) cyclic autocorrelation of $x(t)$

$$R_{\widehat{\boldsymbol{x}}}^{(T)}(\alpha,\tau;t_0) \triangleq \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} \widehat{x}(t+\tau)\,\widehat{x}^{(*)}(t)\,e^{-j2\pi\alpha t}\,\mathrm{d}t$$

Strength of the (conjugate) cyclic correlogram of $\widehat{x}(t)$

$$\lambda_{\widehat{\boldsymbol{x}}}^{(T)}(\alpha) \triangleq \int_{\mathscr{T}} \left| R_{\widehat{\boldsymbol{x}}}^{(T)}(\alpha,\tau;t_0)\right|^2 \mathrm{d}\tau$$

short-time Fourier transform (STFT) of $\widehat{x}(t)$

$$\widehat{X}_Z(t,f) \triangleq \int_{t-Z/2}^{t+Z/2} \widehat{x}(u)\,e^{-j2\pi f u}\,\mathrm{d}u$$

Measurement of Statistical Functions of the Underlying ACS Process (cont'd)

(conjugate) cyclic periodogram of $\widehat{x}(t)$

$$I_{\widehat{\boldsymbol{x}}}^{(T)}(\alpha, f; t_0) \triangleq \frac{1}{T} \widehat{X}_T(t_0, f)\, \widehat{X}_T^{(*)}(t_0, {}_{(-)}(\alpha - f))$$

frequency-smoothed (conjugate) cyclic periodogram of $\widehat{x}(t)$ = estimate of the (conjugate) cyclic spectrum of $x(t)$

$$S_{\widehat{\boldsymbol{x}}}^{(T,\Delta f)}(\alpha, f; t_0) \triangleq \frac{1}{\Delta f} \int_{f-\Delta f/2}^{f+\Delta f/2} I_{\widehat{\boldsymbol{x}}}^{(T)}(\alpha, \lambda; t_0)\, \mathrm{d}\lambda$$

time-smoothed (conjugate) cyclic periodogram of $\widehat{x}(t)$ = estimate of the (conjugate) cyclic spectrum of $x(t)$

$$G_{\widehat{\boldsymbol{x}}}^{(\Delta f,T)}(\alpha, f; t_0) = \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} I_{\widehat{\boldsymbol{x}}}^{(1/\Delta f)}(\alpha, f; t)\, \mathrm{d}t\,.$$

Measurement of Statistical Functions of the Underlying ACS Process (cont'd)

median-filtered frequency-smoothed (conjugate) cyclic periodogram of $\widehat{x}(t)$ = estimate of the 2nd-order cyclic polyspectrum of $x(t)$

$$P_{\widehat{\boldsymbol{x}}}^{(T,\Delta f,\delta f)}(\alpha, f) \triangleq \underset{\nu \in J(f,\delta f)}{\operatorname{med}} \left[S_{\widehat{\boldsymbol{x}}}^{(T,\Delta f)}(\alpha, \nu) \right]$$

inverse Fourier transform of the median-filtered frequency-smoothed (conjugate) cyclic periodogram of $\widehat{x}(t)$ = estimate of the (conjugate) cyclic autocovariance of $x(t)$

$$C_{\widehat{\boldsymbol{x}}}^{(T,\Delta f,\delta f)}(\alpha, \tau) \triangleq \mathscr{F}^{-1} \left[P_{\widehat{\boldsymbol{x}}}^{(T,\Delta f,\delta f)}(\alpha, f) \right]$$

Approximations and Consistency of Estimates

- Angle-modulated sine waves have spectral support covering the entire spectral domain. Thus, the extraction of a single modulated harmonic by filtering is approximate.
- The low-pass filter $h_W(t)$ removes the spectral content at high frequencies of the down-converted modulated sine wave at frequency γ_0 and does not cut the tails of the spectral content of the modulated sine waves at frequencies $\gamma \neq \gamma_0$.
- Consequently, estimates $\widehat{\varepsilon}(t)$ and $\widehat{a}(t)$ are **biased** and so is $\widehat{x}(t)$.
- The interpolation formula to compute $y(t-\widehat{\varepsilon}(t))$ is not exact since $y(t)$ is not strictly band limited. The error can be made arbitrarily small by choosing the sampling period T_s sufficiently small.
- The biasedness of $\widehat{x}(t)$ in general prevents the consistency of the cyclic statistic estimates.
- The larger are $\varepsilon(t)$ and $\dot{\varepsilon}(t)$, the larger is the performance degradation in the estimates.

Numerical Results

Data files: Royal Observatory of Belgium – Solar Influences Data Analysis Center
https://www.sidc.be/SILSO/datafiles

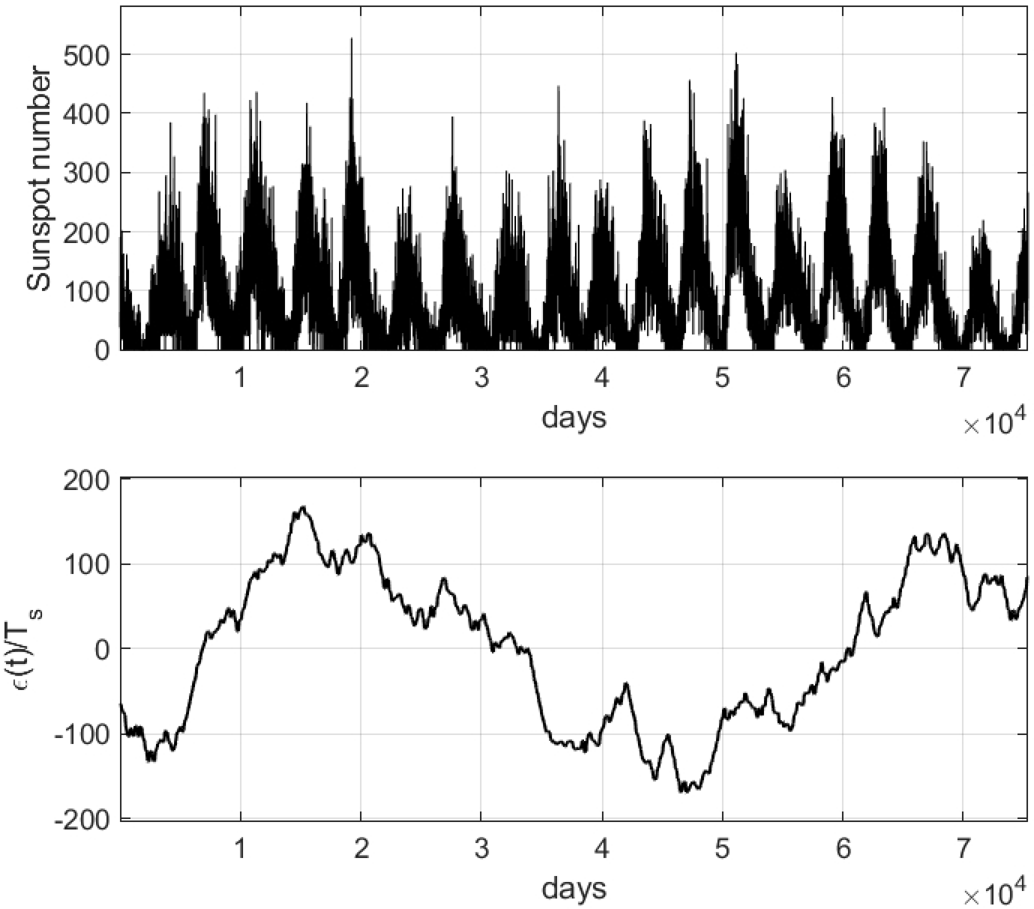
sampling period = 1 day

- 11 years = 11*365 days = 4015 days; $\widetilde{\alpha}_0 = 1/4015 = 0.00025$
- The speed of rotation of the Sun varies according to latitude.
  Rotation period = 28 days at latitude 40 deg; $\widetilde{\alpha}_0 = 1/28 = 0.0357$

sampling period = 1 month

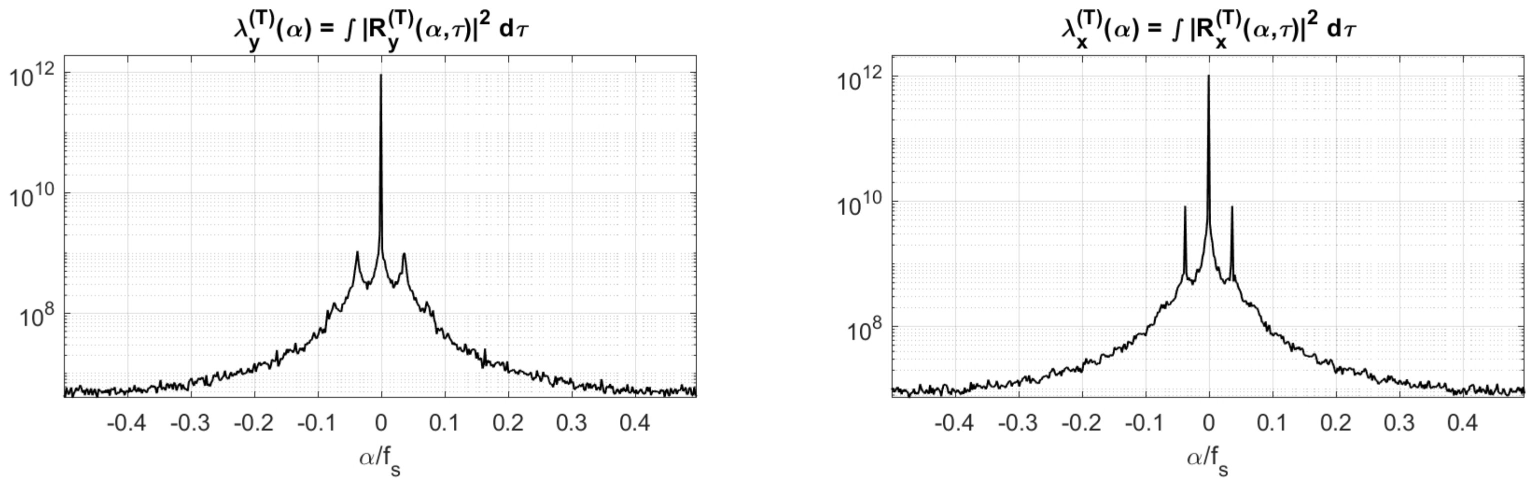
- 11 years = 11*12 months = 132 months; $\widetilde{\alpha}_0 = 1/132 = 0.0076$

27.3-Day Irregular Period



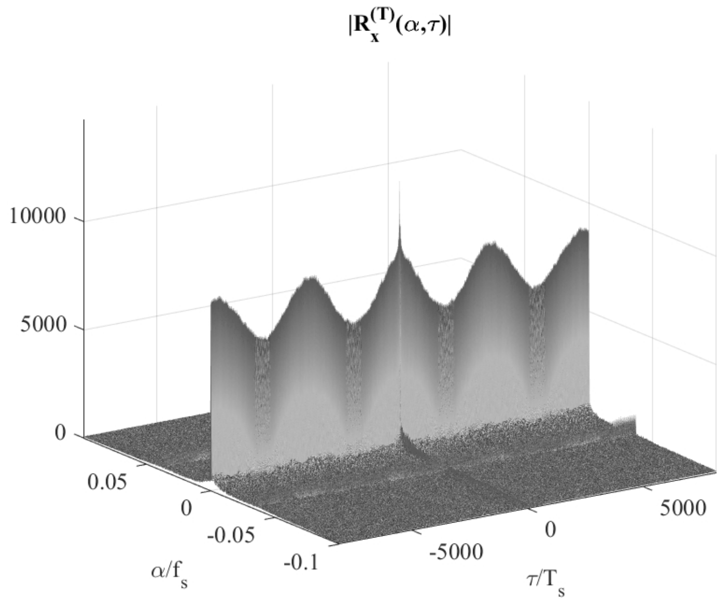
(Top) daily total Sunspot number in the years 1818–2023. (Bottom) estimated time-warping function ($\widetilde{\alpha}_0 \simeq 0.0365\, f_s$, $W = 0.003\, f_s$, $T_s = 1/f_s = 1$ day).

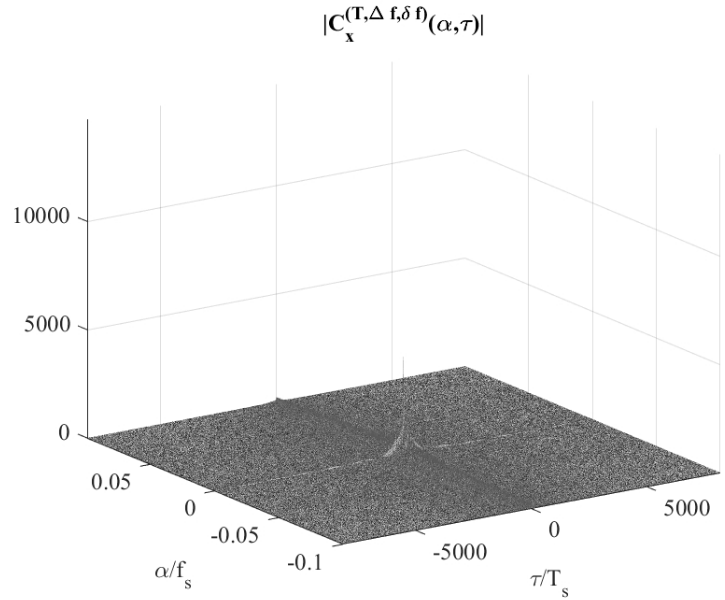
27.3-Day Irregular Period (cont'd)



Strength of the cyclic correlogram as a function of the cycle frequency α. (Left) time series $y(t)$ of the daily total Sunspot number in the years 1818–2023. (Right) de-warped time series $x(t)$.

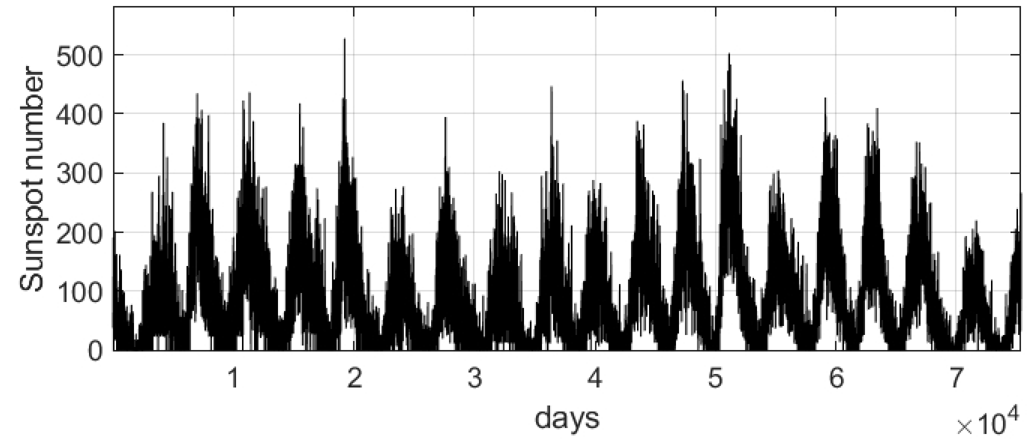
27.3-Day Irregular Period (cont'd)

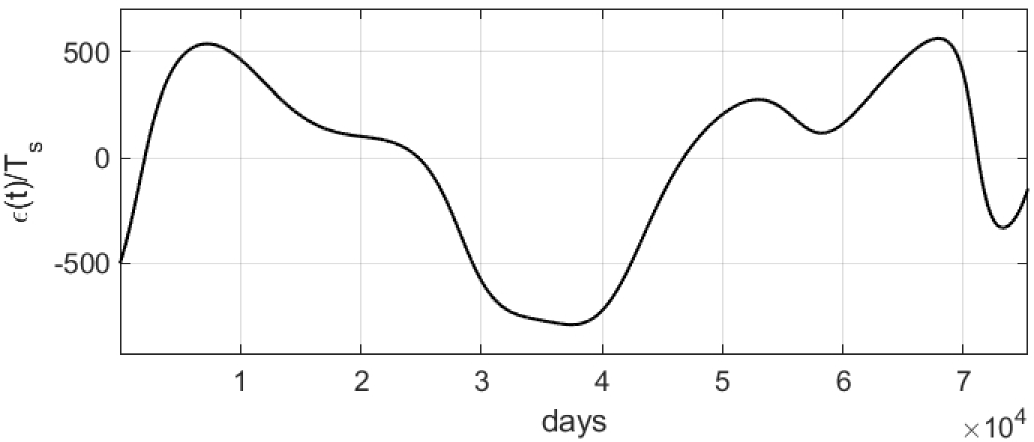




De-warped time series $x(t)$ of the daily total Sunspot number in the years 1818–2023. (Left) Magnitude of the cyclic correlogram as a function of the cycle frequency α and the lag parameter τ. (Right) Magnitude of the estimate of the cyclic autocovariance as a function of α and τ.

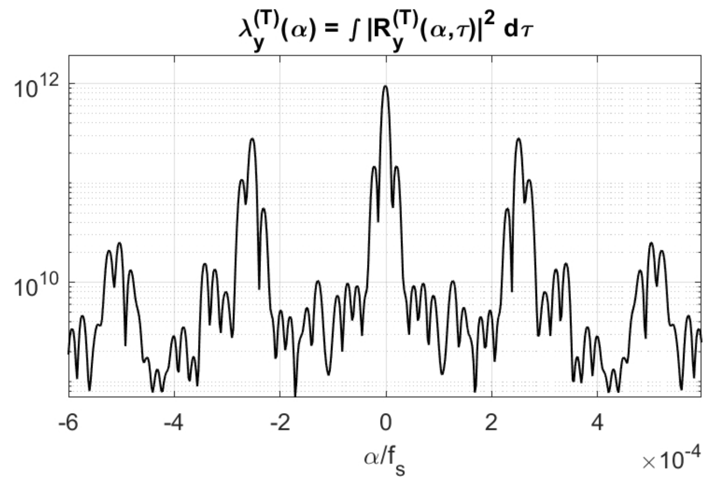
11-Year Irregular Period

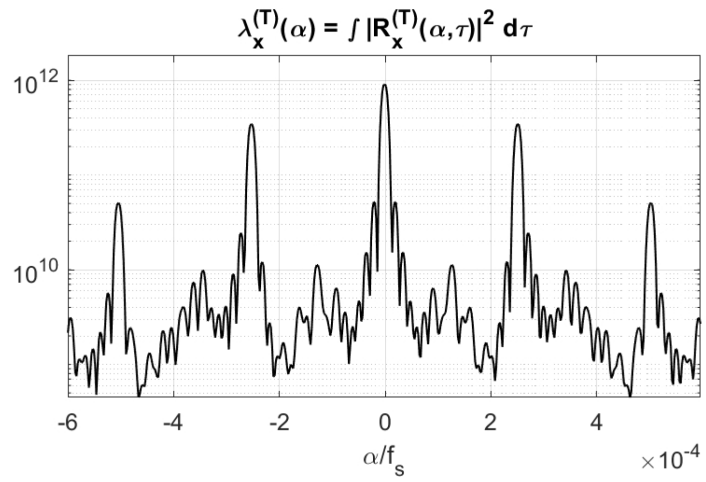




(Top) daily total Sunspot number in the years 1818–2023. (Bottom) Estimated time-warping function ($\widetilde{\alpha}_0 \simeq 0.00024906\, f_s$, $W = 0.00010\, f_s$, $T_s = 1/f_s$ = 1 day).

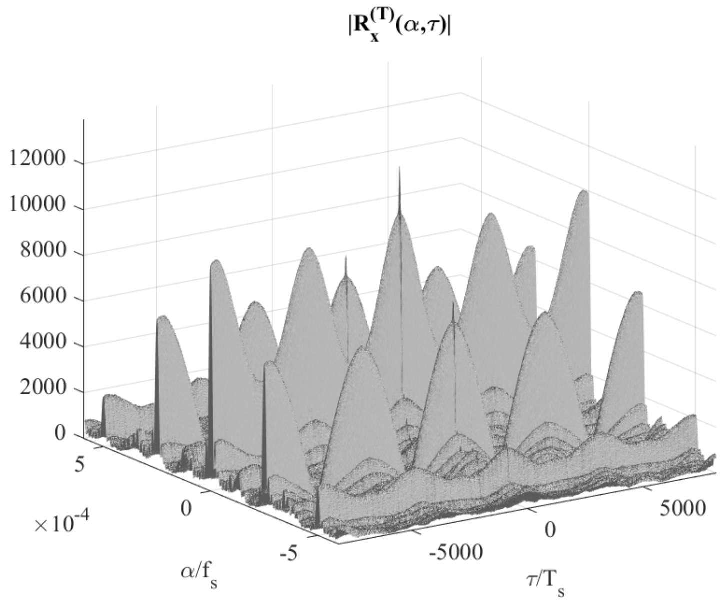
11-Year Irregular Period (cont'd)

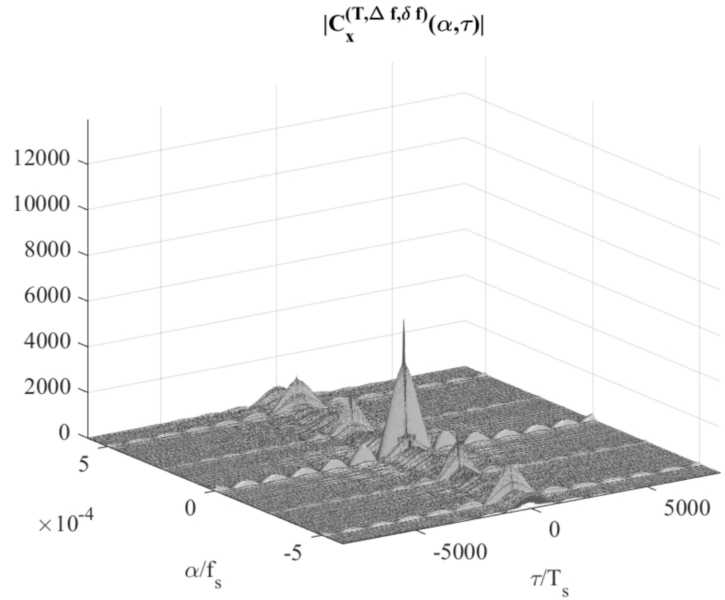




Strength of the zoom in $\alpha \in (-0.0006\, f_s, 0.0006\, f_s)$ of the cyclic correlogram as a function of the cycle frequency α. (Left) time series $y(t)$ of the daily total Sunspot number in the years 1818–2023. (Right) de-warped time series $x(t)$.

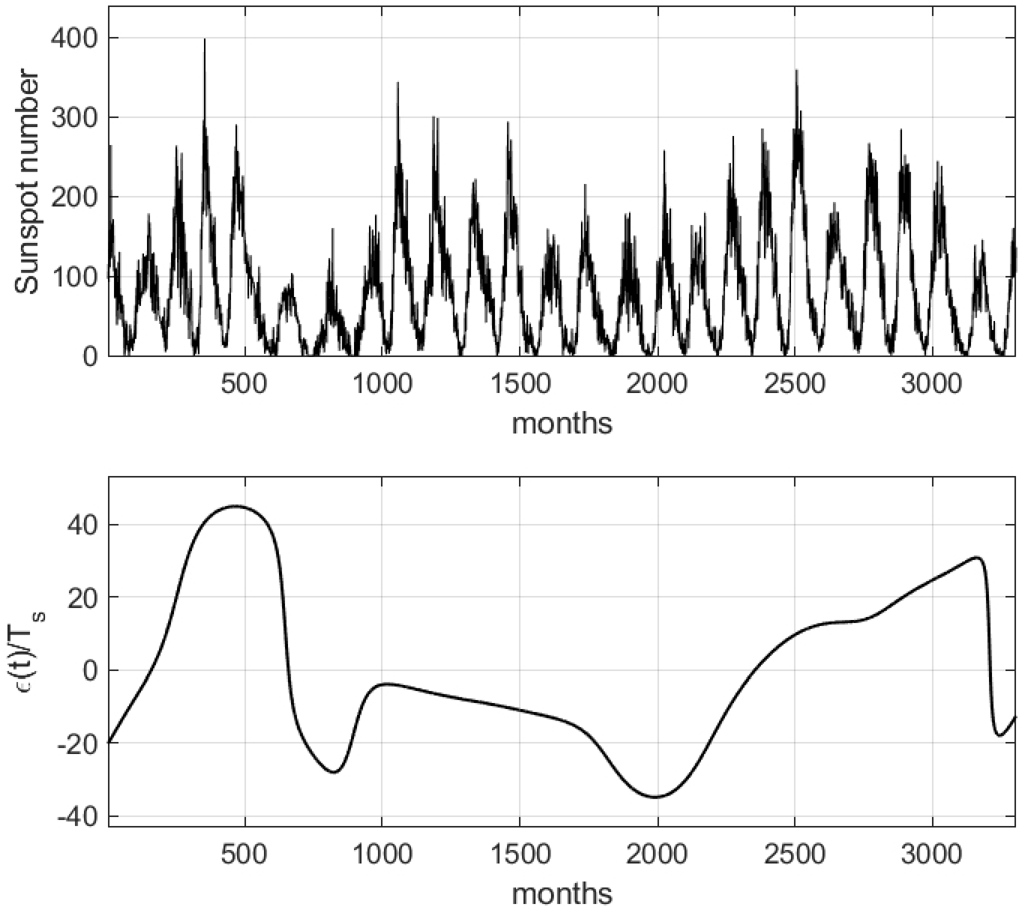
11-Year Irregular Period (cont'd)





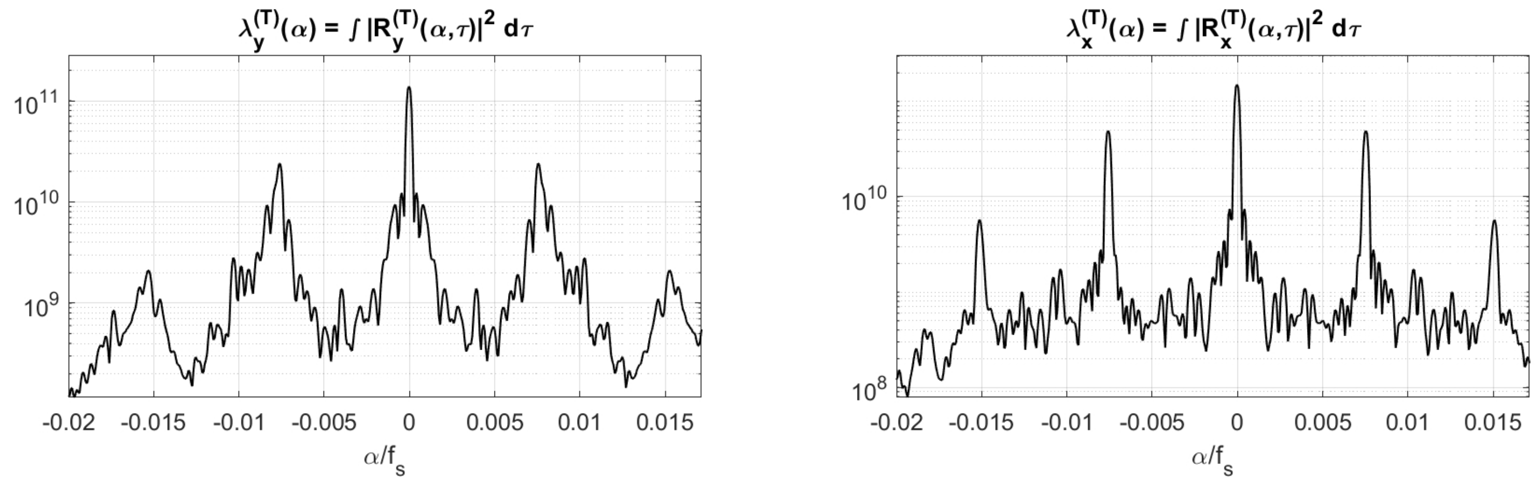
De-warped time series of the daily total Sunspot number in the years 1818–2023. (Left) Magnitude of the cyclic correlogram as a function of the cycle frequency α and the lag parameter τ. (Right) Magnitude of the estimate of the cyclic autocovariance as a function of α and τ. Zoom in $\alpha \in (-0.0006\, f_s, 0.0006\, f_s)$.

11-Year Irregular Period (cont'd)



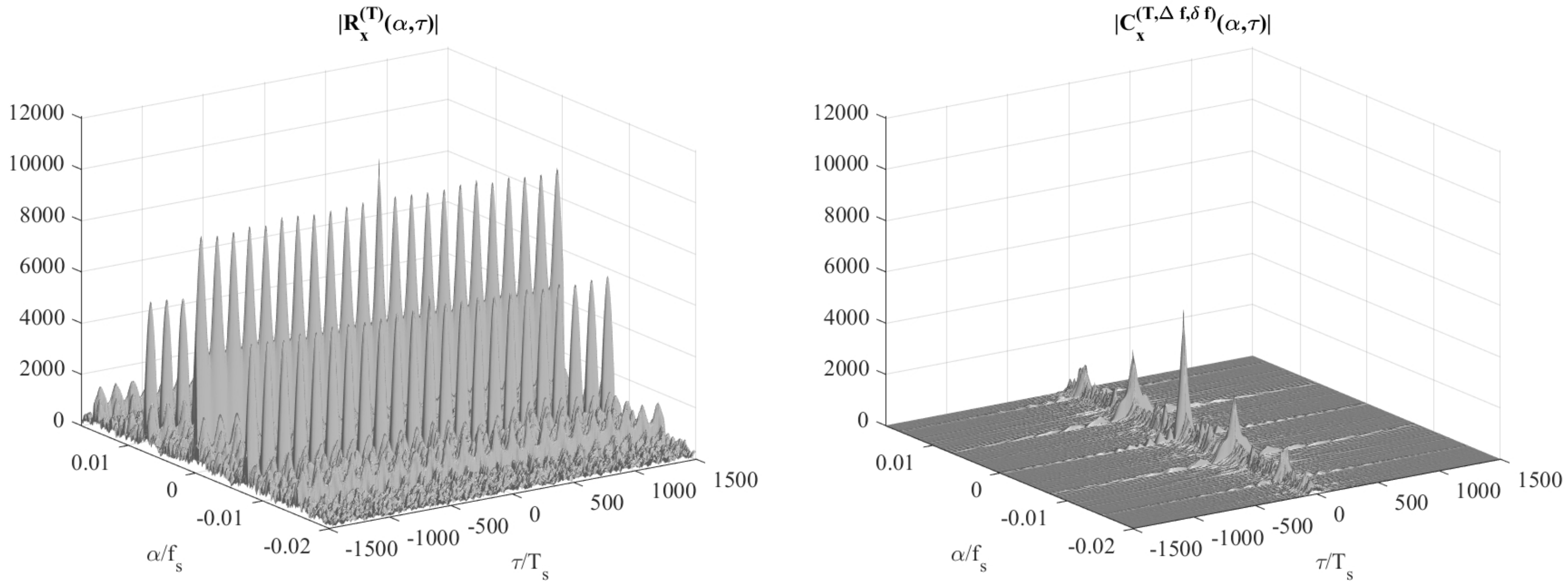
(Top) monthly mean total Sunspot number in the years 1749–2023. (Bottom) estimated time-warping function. ($\widetilde{\alpha}_0 \simeq 0.00753\, f_s$, $W = 0.0030\, f_s$, $T_s = 1/f_s$ = 1 month).

11-Year Irregular Period (cont'd)



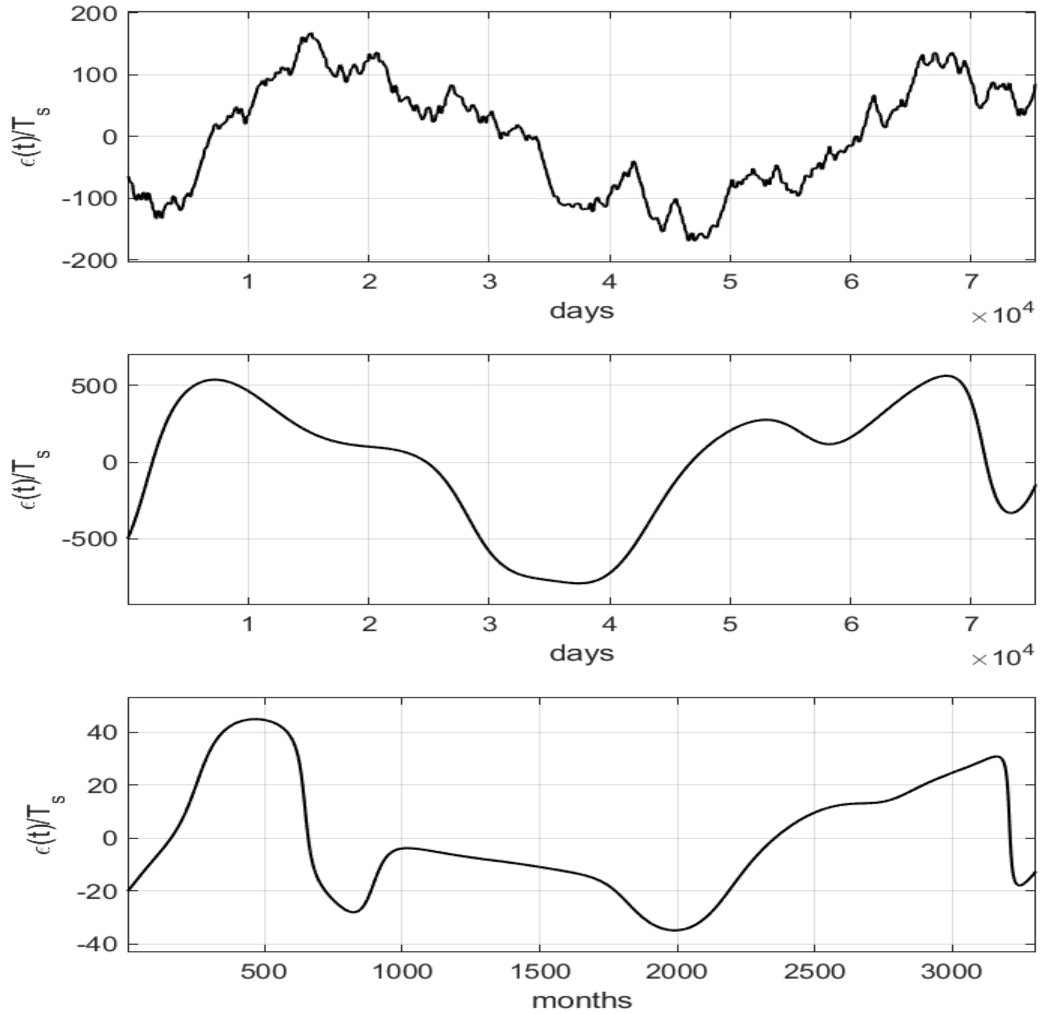
Strength of the cyclic correlogram as a function of the cycle frequency α. (Left) time series $y(t)$ of the monthly mean total Sunspot number in the years 1749–2023. (Right) de-warped time series $x(t)$.

11-Year Irregular Period (cont'd)



De-warped time series of the monthly mean total Sunspot number in the years 1749–2023. (Left) Magnitude of the cyclic correlogram as a function of the cycle frequency α and the lag parameter τ. (Right) Magnitude of the estimate of the cyclic autocovariance as a function of α and τ.

120-200-Year Irregular Periods



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