Harmonizable VARMA time series

Dominique Dehay

IRMAR, Université de Rennes, France
`dominique.dehay@univ-rennes2.fr`

`joint project with A. Dudek, J-M. Freyermuth`

Chapel Hill, March 2025

Abstract

- Harmonizable time series form a wide class of nonstationary time series,
- Tractable Fourier analysis, spectral distributions with correlated components
- Many fields of application, for example, analysis of replicated ElectroEncephaloGram signals for studying the brain connectivity.

  Here
- Parametric form : Harmonizable Vector AutoRegressive and Moving Average models (HVARMA)
- Same spirit as of standard VARMA models: solution of a difference equation
- Harmonizable noise: uncorrelated, but variance not constant over time
- Give their spectral characteristics
- Provide a methodology for generating realizations of harmonizable time series based on known spectral characteristics.
- Discuss how to estimate the parameters on a simple example of an univariate HVARMA (1,0) model.
- Do not assume any model for the harmonizable noise except its heteroskedasticity.

Aim of the talk:

Review and complements on multivariate linear harmonizable models : (Rozanov, 1959), (Rao, 1982), (Mehlman 1991).

Starting point of this review:

Construction and simulation of harmonizable time series with specific spectrum.

EEG and spectral analysis

- Spectral analysis is essential tool for neuroscientists. The spectrum of EEG time series has a direct neurophysiological interpretation.
- Multivariate time series problem: coherence analysis. But with such complex processes, the study of functional connectivity requires dual-frequency coherence.
  (Neuroscience justifies the importance of capturing dependencies between different frequency bands at different locations)
- Dynamic phenomenon
- Several subjects: can we identify differences in subject specific reaction patterns ?
- ...

Spectral analysis: illustration

Consider an univariate time series $(X_n)_{n\in\mathbb{Z}}$.
We observe replicated realizations $\{X_n^r : t = 1, \dots, N; r = 1, \dots, R\}$.
Define its Loève spectrum as

$$\tilde{m}_{X,X} = \frac{1}{4\pi^2 R}\sum_{r=1}^{R} d_X^r(\omega_1)\overline{d_X^r(\omega_2)}, \quad \omega_1, \omega_2 \in [0,_p i)$$

where $d_X^r(\omega)$ is the Fourier transform of the r-th replicate.

- *Fig.1* : ElectroEncephaloGram
- *Fig.2* : Stationary time series
- *Fig.3* : Locally stationary process
- *Fig.4* : PC time series

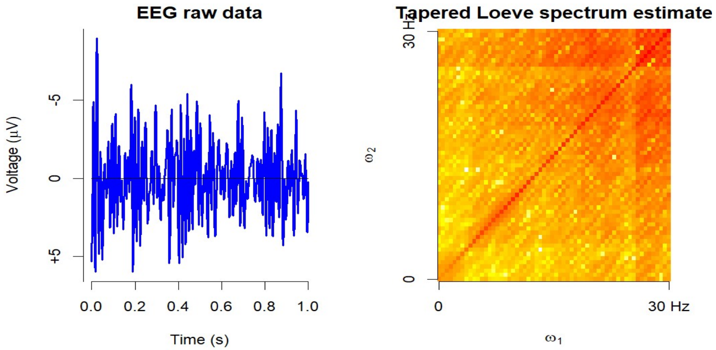


Figure: 1 ElectroEncephalGram

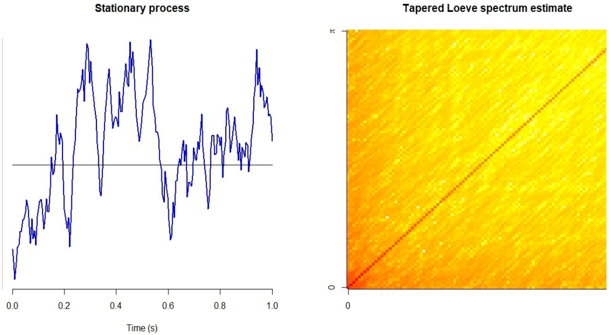


Figure: 2 Stationary time series

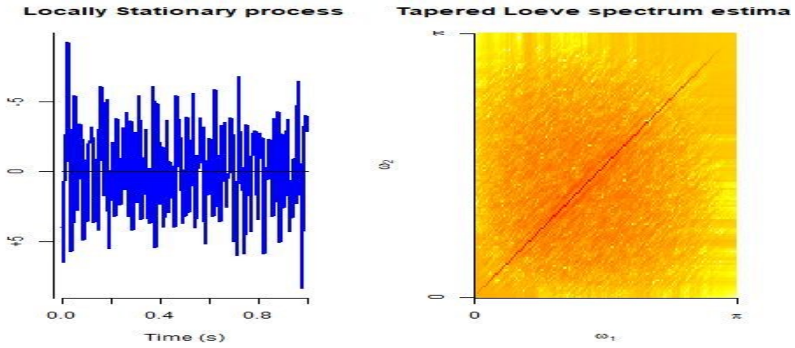
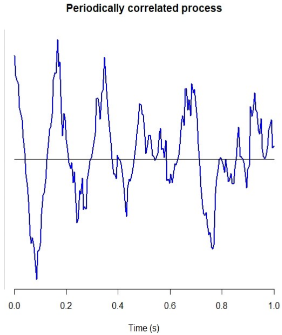


Figure: 3 Locally stationary time series



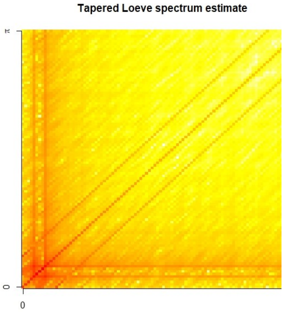


Figure: 4 PC time series

Context

- Second order structure.
- How to capture correlation across different frequencies?
  That is where harmonizable processes comes in (Loève, 1945).

- Consider $(\epsilon_n)_{n\in\mathbb{Z}}$ be a real-valued white noise :

  $$\mathrm{E}(\epsilon) = 0 \quad \text{and} \quad \mathrm{E}(\epsilon_n \epsilon_m) = \mathbb{I}_{\{n=m\}}.$$

  Let $Z_n = \epsilon_n$ for $n \geq 0$ and $Z_n = 0$ for $n < 0$.

  Then $(Z_n)_{n\in\mathbb{N}}$ is not stationary.

  Question: What is the spectral structure of $(Z_n)_{n\in\mathbb{Z}}$?

  Answer: Weakly harmonizable not strongly harmonizable (Abreu 1970).

Context

I – Multivariate harmonizable time series

II – Harmonizable noise

III – Harmonizable linear model

IV – Harmonizable VARMA model

V – Estimation of the parameter of an Harmonizable AR(1) model

Notation

In what follows, we solely consider centered square-integrable d-variate time series and we model their second order structure via frequencies correlation.

For $\mathsf{X} = (X_1, \dots, X_d)^\top$ and $\mathsf{Y} = (Y_1, \dots, Y_d)^\top$ in $\mathcal{H}^d = L^2_{\mathbb{R}^{1\times d}}(\mathrm{P})$

$$\mathrm{Cov}[\mathsf{X}, \mathsf{Y}] = \mathrm{E}\big[\mathsf{X}\mathsf{Y}^\top\big] = \Big(\mathrm{E}[X_j Y_k]\Big)_{j,k=1,\dots,d} \quad \text{and} \quad \mathrm{Var}\,[\mathsf{X}] \stackrel{\text{def}}{=} \mathrm{Cov}\,[\mathsf{X}, \mathsf{X}]$$

$\mathbb{C}^d$ denotes d-dimensional row-vector complex numbers
$\mathbb{C}^{1\times d}$ denotes d-dimensional column vector complex numbers

$\mathbb{T} = \mathbb{R}/2\pi \sim (-\pi, \pi]$

Here, $^\top$ denotes the transpose operator, and * denotes the adjoint operator, i.e. the conjugate transpose operator.

I – Multivariate harmonizable time series

Definition ((Rozanov 1959), (Rao 1982), (Mehlman 1991))

A d-variate time series $(\mathrm{X}_n)_{n\in\mathbb{Z}}$ is said to be harmonizable when it admits a spectral stoch. measure $\boldsymbol{\mu}_{\mathrm{X}} : \mathcal{B}(\mathbb{T}) \to \mathcal{H}^d = L^2_{\mathbb{C}^{1\times d}}(\mathrm{P})$, ($\sigma$-additive) such that

$$\mathrm{X}_n = \int_{\mathbb{T}} e^{in\lambda}\, d\boldsymbol{\mu}_{\mathrm{X}}(\lambda).$$

The integral is in the sense of Dunford-Schwartz, 1957.

Proposition

A centered time series X *is harmonizable if and only if it admits a spectral bimeasure* $\mathrm{M}_{\mathrm{X}} : \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T}) \to \mathbb{C}^{d\times d}$ *(σ-additive w.r.t. each component) such that*

$$\mathrm{cov}[\mathrm{X}_m, \mathrm{X}_n] = \int_{\mathbb{T}}\int_{\mathbb{T}} e^{i(m\lambda_1 - n\lambda_2)}\, d\mathrm{M}_{\mathrm{X}}(\lambda_1, \lambda_2).$$

The above integral is a Morse-Transue integral (Rao, 1982).

- Relationship between spectral stoch. meas. and spectral bimeasure:

$$\mathsf{M}_\mathsf{X}(A,B) = \mathrm{Cov}[\boldsymbol{\mu}_\mathsf{X}(A), \boldsymbol{\mu}_\mathsf{X}(B)] = \mathrm{E}\big[\boldsymbol{\mu}_\mathsf{X}(A)\,\boldsymbol{\mu}_\mathsf{X}(B)^*\big].$$

- The bimeasure M_X is not necessarily extendible as a meas. on $\mathbb{T}^2$.

- When the bimeasure M_X is extendible as a meas. on $\mathbb{T}^2$, the time series is said to be strongly harmonizable, (that is, "harmonizable in the sense of Loève".)

  Otherwise it is called weakly harmonizable (or simply, harmonizable).

- The 2nd order stationary times series are strongly harmonizable.

- A centered strongly harmonizable time series X is stationary
  if and only if its spectral bimeasure M_{X} is concentrated on the first diagonal of $\mathbb{T}^2$:

$$\mathrm{M}_{\mathrm{X}}(A, B) = 0, \qquad \text{for} \quad A \cap B = \emptyset,\ A, B \in \mathcal{B}(\mathbb{T}).$$

That is its stoch. meas. $\boldsymbol{\mu}_{\mathrm{X}}$ is orthogonally scattered:

$$\mathrm{E}\big[\boldsymbol{\mu}_{\mathrm{X}}(A)\boldsymbol{\mu}_{\mathrm{X}}(B)^*\big] = \mathrm{Cov}[\boldsymbol{\mu}_{\mathrm{X}}(A), \boldsymbol{\mu}_{\mathrm{X}}(B)] = 0, \quad \text{when} \quad A \cap B = \emptyset.$$

Then the spectral meas. m_{X} on $\mathbb{T}$ of the stationary time series X:

$$\mathrm{m}_{\mathrm{X}}(A \cap B) = \mathrm{M}_{\mathrm{X}}(A, B) = \mathrm{M}_{\mathrm{X}}(B, A)^*, \qquad \text{for any} \quad A, B \in \mathcal{B}(\mathbb{T}).$$

II – Multivariate harmonizable noise

Recall : we seek for a parametric form for harmonizable time series with an intelligible spectrum.

- Need to ensure the harmonizablility of our ARMA process.
- Proceed alike to the stationary case : define a VARMA harmonizable time series as the unique solution of difference equation based on an harmonizable noise.
- Harmonizable noises to be considered as innovation for harmonizable models.

Main ingredient: "the linear space of weakly harmonizable processes is a module over the class of all bounded linear operators on $\mathcal{H}^d$" (Rao, 1984).

Recall

Definition

A zero-mean d-variate time series $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a <u>d-variate white noise</u> when

$$\mathrm{E}[\varepsilon_m \varepsilon_n^\top] = \mathrm{Cov}[\varepsilon_m, \varepsilon_n] = \Sigma_\varepsilon^2$$

for some symmetrical non-negative-definite $d \times d$-matrix Σ_ε^2.

A d-variate white noise is a stationary time series.

Definition

In this work a <u>d-variate pure white noise</u> is a d-variate white noise such that

$$\Sigma_\varepsilon^2 = \mathbb{I}_{d\times d} \qquad (d \times d\text{-identity matrix}).$$

Lemma (Fundamental lemma)

Let $\mathrm{Z}_n \stackrel{\text{def}}{=} (Z_{n,1}, \ldots, Z_{n,d})^\top$ *such that*

$$\mathrm{E}[\mathrm{Z}_n] = 0_d \quad \textit{and} \quad \mathrm{E}\left[\mathrm{Z}_n\mathrm{Z}_m^\top\right] = \mathrm{Cov}\left[\mathrm{Z}_n, \mathrm{Z}_m\right] = \Sigma^2_{\mathrm{Z}_n}\mathbb{I}_{\{m=n\}}.$$

We have equivalence between:

- <u>*Condition* (Ha)</u>*:* $\sup_{n,j} \mathrm{E}[|Z_{n,j}|^2] < \infty$.
- *There exist a d-variate white noise* $(\varepsilon_n)_{n\in\mathbb{Z}}$ *and a family of* $d \times d$*-matrices* $\mathrm{S}_n = (s_{n,j,k})_{j,k=1,\ldots,d}$ *such that*

$$\sup_{n,j,k} |s_{n,j,k}| < \infty \qquad \textit{and} \qquad \mathrm{Z}_n = \mathrm{S}_n\varepsilon_n, n \in \mathbb{Z}.$$

<u>Then</u> $\mathrm{Var}[\mathrm{Z}_n] = \mathrm{S}_n\Sigma_\varepsilon\mathrm{S}_n^*$.

For a pure white noise $\mathrm{Var}[\mathrm{Z}_n] = \mathrm{S}_n\mathrm{S}_n^*$, $\|\Sigma^2_{\mathrm{Z}_n}\| = \mathrm{E}\left[\mathrm{Z}_n^\top\overline{\mathrm{Z}_n}\right] = \mathrm{tr}\left(\mathrm{S}_n\mathrm{S}_n^*\right)$.

Harmonizability of a non-correlated time series $(\mathrm{Z}_n)_{n\in\mathbb{Z}}$

Condition (**Ha**): $\mathrm{Z}_n \stackrel{\text{def}}{=} (Z_{n,1}, \dots, Z_{n,d})^\top$ such that $\mathrm{E}[\mathrm{Z}_n] = 0_d$

$$\mathrm{E}\left[\mathrm{Z}_n \mathrm{Z}_m^\top\right] = \mathrm{Cov}\left[\mathrm{Z}_n, \mathrm{Z}_m\right] = \Sigma^2_{\mathrm{Z}_n} \mathbb{I}_{\{m=n\}} \quad \text{and} \quad \sup_n \|\Sigma^2_{\mathrm{Z}_n}\| < \infty.$$

Theorem (Definition of harmonizable noise)

A time series $\mathrm{Z} = (\mathrm{Z}_n)_{n\in\mathbb{Z}}$ fulfilling condition (Ha) is harmonizable.

Such a time series Z is called <u>*harmonizable noise*</u>.

Its spectral stoch. meas. $\boldsymbol{\mu}_\mathrm{Z}$ defined on $\mathcal{B}(\mathbb{T})$ verifies

$$\boldsymbol{\mu}_\mathrm{Z}(A) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) \mathrm{Z}_n,$$

the convergence being in $\mathcal{H}^d(\mathrm{Z}) \subset L^2_{\mathbb{C}^{1\times d}}(\mathrm{P})$.

Remarks

- Then the spectral bimeasure M_Z of $(\mathrm{Z}_n)_{n\in\mathbb{Z}}$ is defined on $\mathcal{B}(\mathbb{T})\times\mathcal{B}(\mathbb{T})$ verifies

$$\mathrm{M}_\mathrm{Z}(A,B)=\mathrm{E}[\boldsymbol{\mu}_\mathrm{Z}(A)\boldsymbol{\mu}_\mathrm{Z}^*(B)]=\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}\left(\int_A\int_B e^{-in(\lambda_1-\lambda_2)}d\lambda_1 d\lambda_2\right)\Sigma^2_{\mathrm{Z}_n}.$$

- If $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a white noise, then it is stationary. Hence harmonizable. The spectral stoch. meas. $\boldsymbol{\mu}_\varepsilon$ is orthogonally scattered, that is,

$$\mathrm{M}_\varepsilon(A,B)\stackrel{\mathrm{def}}{=}\mathrm{E}[\boldsymbol{\mu}_\varepsilon(A)\boldsymbol{\mu}_\varepsilon^*(B)]=0_{d\times d}\quad\text{when}\quad A\cap B=\emptyset.$$

  Moreover, it admits a spectral meas. m_ε defined on $\mathcal{B}(\mathbb{T})$:

$$\mathrm{m}_\varepsilon(A\cap B)=\mathrm{M}_\varepsilon(A,B)=\mathrm{E}\left[\boldsymbol{\mu}_\varepsilon(A)\boldsymbol{\mu}_\varepsilon^*(B)\right],\qquad A,B\in\mathcal{B}(\mathbb{T}).$$

- Let Z be a d-variate harmonizable noise.

  Then every function $f \in L^2(\mathbb{T})$ is integrable w.r.t. $\boldsymbol{\mu}_Z$ and

$$\left\| \int_{\mathbb{T}} f(\lambda)\, d\boldsymbol{\mu}_Z(\lambda) \right\|_{\mathcal{H}^d}^2 \leq \frac{\varsigma^2 d}{2\pi} \int_{\mathbb{T}} |f(\lambda)|^2 \, d\lambda$$

  where $\varsigma^2 = \sup_{n,j,k} |s_{n,j,k}|^2$.

  Thus the spectral stoch. meas. $\boldsymbol{\mu}_Z$ is absolutely continuous w.r.t. Lebesgue meas. on $\mathbb{T}$.

  Nevertheless, this does not mean that there exists a an integrable function $\Delta_Z : \mathbb{T} \to \mathcal{H}^d = L^2_{\mathbb{C}^{1\times d}}(\mathrm{P})$ such that

$$\boldsymbol{\mu}_Z(A) = \int_A \Delta_Z(\lambda)\, d\lambda.$$

  Nor that the spectral bimeasure M_Z is extendible to a $\mathbb{C}^{d\times d}$-valued meas. on $\mathcal{B}(\mathbb{T}^2)$.

Examples of harmonizable noises

Let $\varepsilon \stackrel{\text{def}}{=} (\varepsilon_n)_{n\in\mathbb{Z}}$ a d-variate white noise. The following time series are harmonizable noises.

- $\mathrm{Z}_n^{(0)} \stackrel{\text{def}}{=} \varepsilon_n$ for any $n \in \mathbb{Z}$.
- $\mathrm{Z}_n^{(1)} \stackrel{\text{def}}{=} \varepsilon_n$ for $n \geq 0$, and $\mathrm{Z}_n^{(1)} \stackrel{\text{def}}{=} 0$ otherwise (Abreu, 1970)
- $\mathrm{Z}_n^{(2)} \stackrel{\text{def}}{=} \mathrm{S}_n^{(4)}\varepsilon_n$, where $d = 2$ and $\mathrm{S}_n^{(4)} \stackrel{\text{def}}{=} \begin{pmatrix} s_{1,1,n} & s_{1,2,n} \\ s_{2,1,n} & s_{2,2,n} \end{pmatrix}$, with $\sup_{i,j,n} |s_{i,j,n}| < \infty$.

Examples (seq.)

- Let $\mathrm{X} \stackrel{\mathrm{def}}{=} (\mathrm{X}_n)_{n\in\mathbb{Z}}$ be $L^2_{\mathbb{C}^{1\times d}}(\mathrm{P})$-bounded time series.
  Consider the *innovation series* $\mathrm{Z}^{(3)} = (\mathrm{Z}^{(3)}_n)_{n\in\mathbb{Z}}$ defined by

$$Z^{(3)}_{n,j} \stackrel{\mathrm{def}}{=} X_{n,j} - P_{n-1}X_{n,j},$$

  where $P_{n-1}X_{n,j}$ is the one-step prediction of $X_{n,j}$, that is,

  the orthogonal projection of $X_{n,j}$ onto the sub-Hilbert space $\mathcal{H}_{n-1}(\mathrm{X})$ of $L^2_{\mathbb{C}}(\mathrm{P})$ generated by $\{X_{p,k} : p \leq n-1, k = 1, \ldots, d\}$.

Then the series $\mathrm{Z}^{(i)} = (\mathrm{Z}^{(i)}_n)_{n\in\mathbb{Z}}$, $i = 0, \ldots, 3$, are harmonizable noises.

Theorem (Strongly harmonizable noise)

A d-variate harmonizable noise $(\mathrm{Z}_n)_{n\in\mathbb{Z}}$ *is strongly harmonizable* *if and only if*

$$\Sigma^2_{Z_n} = \int_{\mathbb{T}} e^{in\lambda} d\mathrm{m}_{\mathrm{Z}}(\lambda)$$

where m_{Z} *is some* $\mathbb{C}^{d\times d}$*-valued meas. on* $\mathbb{T}$.

- A white noise $(\varepsilon_n)_{n\in\mathbb{Z}}$ is strongly harmonizable and $\mathrm{m}_\varepsilon(A) = \Sigma_\varepsilon \delta_0(A)$ where δ_0 is Dirac meas. at 0.
- Let $\mathrm{G} \in L^1_{\mathbb{C}^{d\times d}}(\mathbb{T})$, the ("convolution") function

$$\mathrm{D}(\lambda) \stackrel{\mathrm{def}}{=} \int_{\mathbb{T}} \mathrm{G}(u)\mathrm{G}(u+\lambda)^* \, du$$

  has positive semi-definite Fourier matrix-coefficients. Thus there is a strongly harmonizable noise Z such that

$$\Sigma^2_{Z_n} = \int_{\mathbb{T}} e^{in\lambda} \mathrm{D}(\lambda) \, d\lambda$$

Existence of a spectral stoch. density

Existence of a strongly measurable Lebesgue-integrable function $\Delta_Z : \mathbb{T} \to \mathcal{H}^d(Z)$ such that

$$Z_n = \int_{\mathbb{T}} e^{in\lambda} \Delta_Z(\lambda)\, d\lambda.$$

Theorem (stoch. density of noise)

The stoch. meas. $\boldsymbol{\mu}_Z$ admits a density w.r.t. Lebesgue meas. on $\mathbb{T}$ if and only if the total energy of Z is finite :

$$\sum_n \mathrm{tr}\Sigma^2_{Z_n} = \sum_n \mathrm{tr}\mathrm{Var}[Z_n] = \sum_n \mathrm{E}\left[Z_n^\top \overline{Z_n}\right] = \sum_n \sum_j \mathrm{E}(Z^2_{n,j}) < \infty.$$

In this case, the time series Z is strongly harmonizable: the spectral bimeasure M_Z is extensible to a complex meas. on $\mathbb{R}^2$ with density

$$F_Z(\lambda_1, \lambda_2) = D_Z(\lambda_1 - \lambda_2) \stackrel{\mathrm{def}}{=} \frac{1}{4\pi^2} \sum_{n\in\mathbb{Z}} e^{-in(\lambda_1 - \lambda_2)} \Sigma^2_{Z_n}.$$

$$\Delta_Z(\lambda) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-in\lambda} Z_n, \quad \textit{the convergence being in } L^2_{\mathbb{C}^d}(\mathrm{P}).$$

Existence of a spectral stoch. density (seq.)

In this case, the spectral bimeasure M_Z is extendible to a $\mathbb{C}^{d\times d}$-valued meas. on $\mathbb{T}^2$ with density

$$\mathrm{E}[\Delta_\mathsf{Z}(\lambda_1)\Delta^*_\mathsf{Z}(\lambda_2)] = \mathsf{D}_\mathsf{Z}(\lambda_1-\lambda_2)$$

w.r.t. Lebesgue meas. on $\mathbb{T}^2$, and where

$$\mathsf{D}_\mathsf{Z}(\lambda) = \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}} e^{-in\lambda}\Sigma^2_{\mathsf{Z}_n}.$$

The spectral stoch. density Δ_Z is stationnary w.r.t. the frequencies. Furthermore

$$\Sigma^2_{\mathsf{Z}_n} = \int_\mathbb{T} e^{i\lambda n}\, d(\mathsf{m}_\mathsf{Z}\lambda) = \int_\mathbb{T} e^{i\lambda n}\mathsf{D}_\mathsf{Z}(\lambda)\, d\lambda.$$

Notice that a (non-null) white noise $(\varepsilon_n)_{n\in\mathbb{Z}}$ has no spectral stoch. density : Indeed, condition $\sum_n \mathrm{tr}\Sigma^2_{\varepsilon_n} < \infty$ is not fulfilled since $\Sigma^2_{\varepsilon_n} = \mathit{constante}$.

III – Multivariate harmonizable linear model

Stability by application of a finite time-invariant linear filter on an harmonizable time series.

Theorem (Harmonizable linear model)

Let $\mathrm{Z} = (Z_n)_{n\in\mathbb{Z}}$ be a d-variate harmonizable noise and $(\mathrm{A}_n)_{n\in\mathbb{Z}}$ be a family of complex $d \times d$-matrices with $\sum_n \mathrm{tr}(\mathrm{A}_n \mathrm{A}_n^*) < \infty$. <u>Then</u> the d-variate time series $\mathrm{X} = (\mathrm{X}_n)_{n\in\mathbb{Z}}$

$$\mathrm{X}_n = \sum_{m\in\mathbb{Z}} \mathrm{A}_{n-m} \mathrm{Z}_j = \sum_{m\in\mathbb{Z}} \mathrm{A}_m \mathrm{Z}_{n-m}, \quad \text{(convergence in } \mathcal{H}(\mathrm{Z}) \subset L^2_{\mathbb{C}^d}(\mathrm{P})\text{)}$$

is well defined and harmonizable.

Moreover,

If Z is strongly harmonizable <u>then</u> X is strongly harmonizable.

If Z has a spectral stoch. density <u>then</u> X has a spectral stoch. density.

If Z has a discrete spectral stoch. meas. <u>then</u> X has a discrete spectral stoch. meas..

Theorem (Harmonizable linear model, seq.)

Moreover the Fourier transform $\widehat{\mathrm{A}} : \mathbb{T} \to \mathbb{C}^{d\times d}$ *is* $\boldsymbol{\mu}_{\mathrm{Z}}$*-integrable and the spectral stoch. meas.* $\boldsymbol{\mu}_{\mathrm{X}}$ *verifies for* $A \in \mathcal{B}(\mathbb{T})$

$$\boldsymbol{\mu}_{\mathrm{X}}(A) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \left(\int_A e^{-in\lambda} \widehat{\mathrm{A}}(-\lambda) d\lambda \right) \mathrm{Z}_n = \int_A \widehat{\mathrm{A}}(-\lambda)\, d\boldsymbol{\mu}_{\mathrm{Z}}(\lambda).$$

Thus

$$\mathrm{X}_n = \int_{\mathbb{T}} e^{in\lambda}\, \widehat{\mathrm{A}}(-\lambda)\, d\boldsymbol{\mu}_{\mathrm{Z}}(\lambda),$$

The spectral bimeasure of X *can be expressed by*

$$\mathrm{M}_{\mathrm{X}}(A, B) = \frac{1}{4\pi^2} \sum_{n\in\mathbb{Z}} \left(\int_A e^{-in\lambda} \widehat{\mathrm{A}}(-\lambda) d\lambda \right) \Sigma^2_{\mathrm{Z}_n} \left(\int_B e^{in\lambda} \widehat{\mathrm{A}}^*(-\lambda) d\lambda \right).$$

Remarks

(1) If Z is strongly harmonizable then X is strongly harmonizable and

$$\mathrm{Cov}[\mathsf{X}_n, \mathsf{X}_m] = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{in((u+\lambda)-m\lambda)} \widehat{\mathsf{A}}(-\lambda + u) \widehat{\mathsf{A}}^*(-\lambda)\, d\lambda\, d\mathsf{m}_{\mathsf{Z}}(u)$$

The spectral bimeasure M_{X} is extendible to a meas. on $\mathbb{T}^2$ and verifies

$$\mathsf{M}_{\mathsf{X}}(A, B) = \frac{1}{2\pi} \int_B \left(\int_{A-\lambda_2} \widehat{\mathsf{A}}(\lambda_1 - \lambda_2) \widehat{\mathsf{A}}^*(-\lambda_2)\, d\mathsf{m}_{\mathsf{Z}}(\lambda_1) \right) d\lambda_2.$$

(2) If Z has a spectral stoch. density <u>then</u> X has a spectral stoch. density $\Delta_X(\lambda) = \widehat{A}(-\lambda)\Delta_Z(\lambda)$:

$$X_n = \int_{\mathbb{T}} e^{in\lambda}\widehat{A}(-\lambda)\Delta_Z(\lambda)d\lambda.$$

Thus

$$\begin{aligned}\mathrm{Cov}\{X_n, X_m\} &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} e^{ij\lambda} A_{n-j} D_Z(\lambda) A^*_{m-j} \right) d\lambda \\ &= \iint_{\mathbb{T}^2} e^{-i(n\lambda_1 - m\lambda_2)}\widehat{A}(-\lambda_1) D_Z(\lambda_1 - \lambda_2)\widehat{A}^*(-\lambda_2)\, d\lambda_1 d\lambda_2.\end{aligned}$$

Recall

$$\Delta_Z(\lambda) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in\lambda} Z_n, \quad D_Z(\lambda) = \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}} e^{-in\lambda}\Sigma^2_{Z_n}, \quad \mathrm{E}[\Delta_Z(\lambda_1)\Delta^*_Z(\lambda_2)] = D_Z(\lambda_1 - \lambda_2)$$

IV – Harmonizable VARMA model

A difference equation on the harmonizable noise Z

$$\mathrm{X}_n + \boldsymbol{\phi}_1 \mathrm{X}_{n-1} + \cdots + \boldsymbol{\phi}_p \mathrm{X}_{n-p} = \mathrm{Z}_n + \boldsymbol{\theta}_1 \mathrm{Z}_{n-1} + \cdots + \boldsymbol{\theta}_q \mathrm{Z}_{n-q}. \qquad (1)$$

where $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_p, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_q \in \mathbb{C}^{d\times d}$. Let

$$\Phi(z) \stackrel{\text{def}}{=} \mathbb{I}_{d\times d} + \boldsymbol{\phi}_1 z + \cdots + \boldsymbol{\phi}_p z^p \quad \text{and} \quad \Theta(z) \stackrel{\text{def}}{=} \mathbb{I}_{d\times d} + \boldsymbol{\theta}_1 z + \cdots + \boldsymbol{\theta}_p z^p.$$

We assume that $\det \Phi(e^{i\lambda}) \neq 0$ for any λ.

Then equat. (1) admits a unique harmonizable solution :

$$\mathrm{X}_n = \int_{\mathbb{T}} e^{in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda})\, d\boldsymbol{\mu}_{\mathrm{Z}}(\lambda)$$

The spectral stoch. meas. $\boldsymbol{\mu}_{\mathrm{X}}$ verifies

$$\boldsymbol{\mu}_{\mathrm{X}}(A) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \left(\int_A e^{-in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda})\, d\lambda \right) \mathrm{Z}_n.$$

- If Z is a strongly harmonizable noise then X is strongly harmonizable.

- If the harmonizable noise Z is of finite energy, then Z and X has spectral stoch. densities and Loève densities F_Z and F_X w.r.t. Lebesgue meas. on $\mathbb{T}^2$

$$\mathrm{Cov}[X_n, X_m] = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - m\lambda_2)} F_X(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$

where

$$F_Z(\lambda_1, \lambda_2) = D_Z(\lambda_1 - \lambda_2), \qquad D_Z(\lambda) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \Sigma^2_{Z_n}$$

and

$$F_X(\lambda_1, \lambda_2) = \Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1}) D_Z(\lambda_1 - \lambda_2) \Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}.$$

Remarks

- Causality (Brockwell & Davis, Theorem 3.1.1) :

$$\det \Phi(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \Longrightarrow \quad X_n = \sum_{j=-\infty}^{n} A_{n-j} Z_j = \sum_{j=0}^{\infty} A_j Z_{n-j}.$$

- Invertibility (Brockwell & Davis, Theorem 3.1.2) :

$$\det \Theta(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \Longrightarrow \quad Z_n = \sum_{j=-\infty}^{n} \Pi_{n-j} X_j = \sum_{j=0}^{\infty} \Pi_j X_{n-j}.$$

Here $\Theta(z)^{-1}\Phi(z) = \sum_{j\in\mathbb{Z}} \Pi_j z^j$

- In the two previous relations, we have the equivalence whenever the polynomial functions $\det \Phi(\cdot)$ and $\det \Theta(\cdot)$ have no common zeroes.

Example : $d = 1$ Let $Z = (Z_n)_{n\in\mathbb{Z}}$, be a univariate harmonizable noise. Consider the difference equation

$$X_n = \phi X_{n-1} + Z_n + \theta Z_{n-1}.$$

- If $|\phi| \neq 1$ Then there exists a unique harmonizable solution

$$X_n = \int_{\mathbb{T}} e^{in\lambda} \frac{1+\theta e^{-i\lambda}}{1-\phi e^{-i\lambda}}\, d\mu_Z(\lambda).$$

- If $\theta = 0$ and $0 < |\phi| < 1$ then

$$X_n = \sum_{j=-\infty}^{n} \phi^{n-j} Z_j = \sum_{j=0}^{\infty} \phi^j Z_{n-j}$$

- If $\theta = 0$ and $|\phi| > 1$ then

$$X_n = \sum_{j=1}^{\infty} (-1)^j \phi^{-j} Z_{n+j}.$$

Construction of an harmonizable time series

Let $N > 0$ be fixed and $\omega_j \stackrel{\text{def}}{=} \frac{2\pi j}{2N+1}$, $j = -N, \ldots, N$.
Let

$$Z_n = \begin{cases} \sum_{j=-N}^{N} e^{in\omega_j} \boldsymbol{\mu}_j & \text{for} \quad n = -N, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

where $(\boldsymbol{\mu}_j)_{j\in\mathbb{Z}}$ is a periodic 2nd order stationary sequence on $\mathbb{Z}$.

Let $K_{\boldsymbol{\mu}}(r) \stackrel{\text{def}}{=} \mathrm{Cov}[\boldsymbol{\mu}_{j+r}, \boldsymbol{\mu}_j]$ for any $j, r \in \mathbb{Z}$.

Then $\boldsymbol{\mu}_{j+2N+1} = \boldsymbol{\mu}_j$ and $K_{\boldsymbol{\mu}}(r + 2N + 1) = K_{\boldsymbol{\mu}}(r)$.

<u>Then</u> $Z = (Z_n)_{n\in\mathbb{Z}}$ is an harmonizable noise with a spectral stoch. density.

The density of the spectral bimeasure M_Z is equal to

$$F_Z(\lambda_1, \lambda_2) = D_Z(\lambda_1 - \lambda_2), \qquad D_Z(\lambda) = \frac{2N+1}{4\pi^2} \sum_{j=-N}^{N} W_N(\omega_j - \lambda) K_{\boldsymbol{\mu}}(j)$$

where $\quad W_N(\lambda) \stackrel{\text{def}}{=} \sum_{n=-N}^{N} e^{in\lambda} = \frac{\sin((2N+1)\frac{\lambda}{2})}{\sin(\frac{\lambda}{2})}$ Dirichlet kernel .

Actually the random variables $\boldsymbol{\mu}_j$ forms a d-variate mean-zero stationary time series on $\mathbb{Z}_N = \mathbb{Z}/(2N+1) \sim \{-N, \ldots, N\}$

Construction (seq.)

Then the harmonizable VARMA time series X defined by

$$\mathrm{X}_n + \phi_1 \mathrm{X}_{n-1} + \cdots + \phi_p \mathrm{X}_{n-p} = \mathrm{Z}_n + \boldsymbol{\theta}_1 \mathrm{Z}_{n-1} + \cdots + \boldsymbol{\theta}_q \mathrm{Z}_{n-q}.$$

is strongly harmonizable with bispectre

$$\mathrm{M}_{\mathrm{X}}(d\lambda_1, d\lambda_2) = \mathrm{F}_X(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$

$$\mathrm{F}_X(\lambda_1, \lambda_2) = \Phi(e^{-i\lambda_1})^{-1}\Theta(e^{-i\lambda_1})\mathrm{D}_{\mathrm{Z}}(\lambda_1 - \lambda_2)\Theta^*(e^{-i\lambda_2})\Phi^*(e^{-i\lambda_2})^{-1}$$

$$= \frac{2N+1}{4\pi^2} \sum_{j=-N}^{N} W_N(\omega_j - \lambda_1 + \lambda_2)\, \Phi(e^{-i\lambda_1})^{-1}\Theta(e^{-i\lambda_1})\mathrm{K}_{\boldsymbol{\mu}}(j)\Theta^*(e^{-i\lambda_2})\Phi^*(e^{-i\lambda_2})^{-1}.$$

Spectral representation

$$\mathrm{X}_n = \frac{1}{2\pi} \sum_{j=-N}^{N} \left(\int_{\mathbb{T}} e^{i\lambda n} \Phi(e^{-i\lambda})^{-1}\Theta(e^{-i\lambda}) W_N(\omega_j - \lambda)\, d\lambda \right) \boldsymbol{\mu}_j$$

$$= \int_{\mathbb{T}} e^{i\lambda n} \Phi(e^{-i\lambda})^{-1}\Theta(e^{-i\lambda}) \left(\frac{1}{2\pi} \sum_{j=-N}^{N} W_N(\omega_j - \lambda)\, \boldsymbol{\mu}_j \right) d\lambda.$$

Spectral analysis: illustration 2

We observe R replications of realizations of two time series X and Y.
Loève cross-spectrum $\tilde{m}_{X,Y}$ and dual-frequency coherence $\tilde{\rho}_{X,Y}$

$$\tilde{m}_{X,Y}(\omega_1,\omega_2) = \frac{1}{4\pi^2 R}\sum_{r=1}^{R} d_X^r(\omega_1)\overline{d_X^r(\omega_2)}$$

$$\tilde{\rho}_{X,Y}(\omega_1,\omega_2) = \frac{|\tilde{m}_{X,Y}(\omega_1,\omega_2)|^2}{\tilde{m}_{X,X}(\omega_1,\omega_1)\tilde{m}_{Y,Y}(\omega_2,\omega_2)}, \quad \omega_1,\omega_2 \in [0,_p i)$$

where $d_X^r(\omega)$ is the Fourier transform of the r-th replicate.

- *Fig.5* : VARMA versus HVARMA (realizations)
- *Fig.6* : HVARMA(1,1)-(2,82) (Loève cross-spect. and Dual-freq. coh.)
- *Fig.7* : VARMA versus HVARMA (realizations)
- *Fig.8* : HVARMA(3,2)-(2,82) (Loève cross-spect. and Dual-freq. coh.)

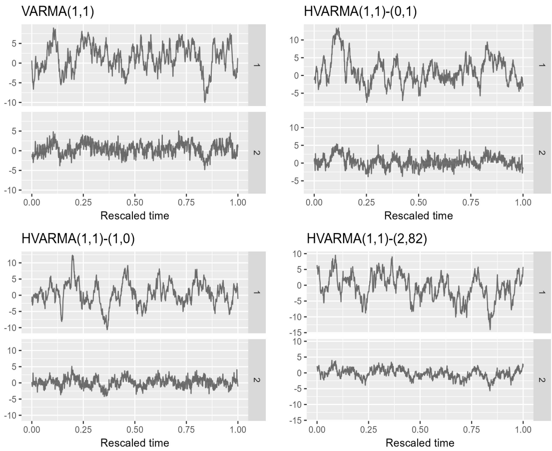


Figure: 5 VARMA versus HVARMA

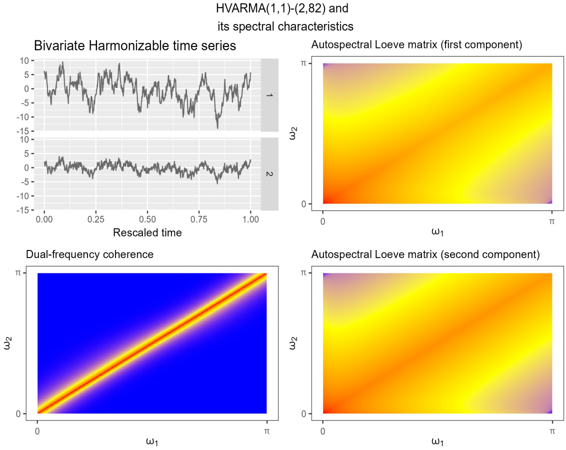


Figure: 6 HVARMA(1,1)(2,82)

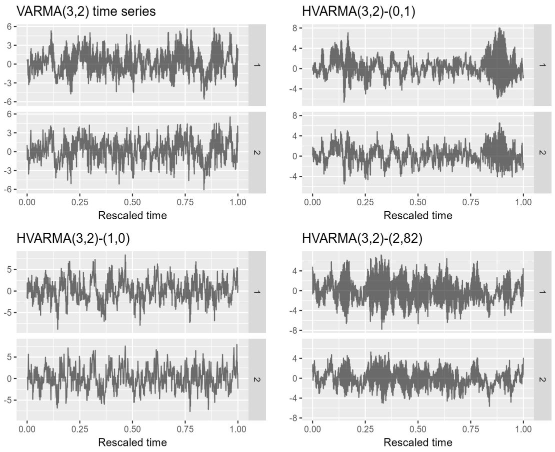


Figure: 7 VARMA versus HVARMA

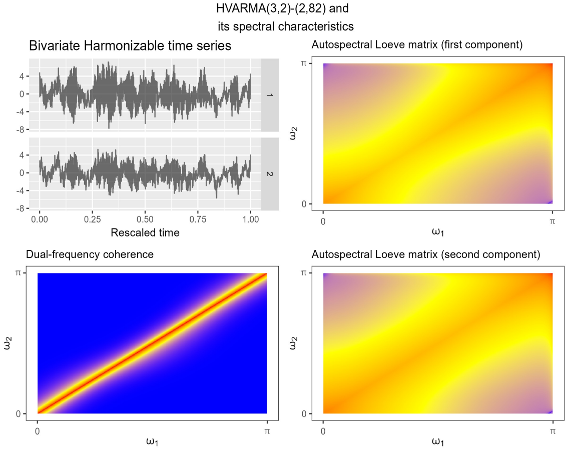


Figure: 8 HVARMA(3,2)(2,82)

VI – Parameter estimation, harmonizable AR(1), $d = 1$

H.B. Mann and A. Wald (1943), T.W. Anderson (1959), J.S. White (1958)

$$X_n - \phi X_{n-1} = Z_n.$$

X_0 and Z_n, $n \geq 1$, are square-integrable, $\mathrm{E}(Z_n) = \mathrm{E}(X_0) = 0$, and

$$Z_n \perp\!\!\!\perp Z_k, \qquad Z_n \perp\!\!\!\perp X_0 \qquad \text{for } n \neq k,$$

where $\perp\!\!\!\perp$ denotes the independence of the random variables.

Then
$$S_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} \phi^{-k} Z_k = (\phi^{-n} X_n - X_0) \quad \perp\!\!\!\perp X_0, Z_k, k \leq n-1$$

and
$$\mathrm{var}(X_n) = \mathrm{E}(X_n^2) = \phi^{2n} \sum_{k=1}^{n} \phi^{-2k} \mathrm{E}(Z_k^2) + \phi^{2n} \mathrm{E}(X_0^2).$$

Notice that, if $|\phi| > 1$ and $\sup_n \mathrm{E}(Z_n^2) < \infty$, then $\lim_{n\to\infty} S_n = S_\infty$ and

$$\lim_{n\to\infty} \phi^{-n} X_n = S_\infty + X_0 \quad \text{in q.m. and a.e}$$

where $\mathrm{E}(S_\infty) = 0$ and $\mathrm{E}(S_\infty^2) = \sum_n \phi^{-2n} \sigma_n^2 < \infty$.

Now we define the least squares estimator of $\phi \in \mathbb{R}$ as follows

$$\widehat{\phi}_n \stackrel{\text{def}}{=} \arg\min_{\phi} \sum_{k=1}^{n} \left(X_k - \phi X_{k-1}\right)^2 = \frac{\sum_{k=1}^{n} X_{k-1} Z_k}{\sum_{k=1}^{n} X_{k-1}^2}.$$

Proposition (Parameter estimation: consistency)

Let $\sup_n \mathrm{E}(Z_n^2) < \infty$, $\sup_n \mathrm{E}(Z_n^4) < \infty$ *and* $\mathrm{E}(X_0^4) < \infty$. *Assume that*

(i) when $|\phi| \leq 1$, $\liminf_n \frac{1}{n} \sum_{k=1}^{n} \mathrm{E}(Z_n^2) > 0$,

(ii) when $|\phi| > 1$, $\mathrm{P}(S_\infty + X_0 = 0) = 0$.

Then

$$\lim_{n\to\infty} \widehat{\phi}_n = \phi \text{ a.e.}$$

Recall for $|\phi| > 1$, $\quad S_\infty \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \phi^{-k} Z_k \quad$ q.m. and a.e.

Strong law of large numbers for martingales (Hall & Heyde 1980, Theorem 2.18)

Proposition (Parameter estimation: $|\phi| < 1$ - Asymptotic normality)

Let $|\phi| < 1$, $\sup_n \mathrm{E}(Z_n^2) < \infty$, $\sup_n \mathrm{E}(Z_n^4) < \infty$, $\mathrm{E}(X_0^4) < \infty$. *In addition, assume that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{E}(Z_j^2) = \sigma^2 > 0 \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-k} \mathrm{E}(Z_j^2)\,\mathrm{E}(Z_{j+k}^2) = \eta_k^4 > 0 \quad \textit{for any } k \geq 1.$$

Then

$$\sqrt{n}\big(\widehat{\phi}_n - \phi\big) \xrightarrow{\textit{in law}} \mathcal{N}(0, \Gamma_\phi)$$

where

$$\Gamma_\phi = (1-\phi^2)^2 \sigma^{-4} \sum_k \phi^{2(k-1)} \eta_k^4.$$

Proposition (Parameter estimation: $|\phi| = 1$ - asymptotic law)

Assume that $|\phi| = 1$, $\sup_n |\mathrm{E}(Z_n^p)| < \infty$ *for any* $p \geq 2$, $\mathrm{E}(X_0^4) < \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{E}(Z_k^2) = \sigma^2 > 0.$$

Then

$$n\left(\widehat{\phi}_n - \phi\right) \xrightarrow{\text{in law}} \frac{B_1^2 - 1}{2\int_0^1 B_u^2\, du} \overset{\text{law}}{=} \frac{\int_0^1 B_u\, dB_u}{\int_0^1 B_u^2\, du}.$$

Here $\{B_t : t \in [0,1]\}$ *be a Brownian motion.*

Lemma (Parameter estimation: $|\phi| > 1$ - particular case - asymptotic law)

Let $|\phi| > 1$, $Z_n = \sigma_n \zeta_n$ *with* $\zeta_n \perp\!\!\!\perp X_0$, $\mathcal{L}(\zeta_n) = \mathcal{L}(\zeta)$

for some random variable ζ *such* $\mathrm{E}(|\zeta|^p) < \infty$ *for any* $p \geq 1$ *and* $\mathrm{E}(\zeta) = 0$.

Assume in addition that $\lim_{n\to\infty} \sigma_n = \varsigma < \infty$.

<u>Then</u> conditioned by $\{\varsigma\zeta_2 + X_0 \neq 0\}$

$$\lim_{n\to\infty} \mathcal{L}\big(\phi^n(\widehat{\phi}_n - \phi)\big) = \mathcal{L}\Big(\frac{(\phi^2 - 1)\varsigma\,\zeta_1}{\varsigma\,\zeta_2 + X_0}\Big)$$

where $\mathcal{L}(\zeta_1) = \mathcal{L}(\zeta_2) = \mathcal{L}(\zeta)$ *and* ζ_1, ζ_2, X_0 *are independent.*

When $X_0 = 0$ and ζ Gaussian, then the law of limit is Cauchy distribution.

Proposition (Parameter estimation: $|\phi| > 1$ - Convergence in law of the mean)

If $\sup_n |\mathrm{E}(Z_n^p)| < \infty$ *for any* $p \geq 1$ *and* $\varsigma^2 \stackrel{\text{def}}{=} \lim_n n^{-1} \sum_{k=1}^n \sigma_k^2$ *exists and is finite.*

Then conditioned by $\{S_\infty + X_0 \neq 0\}$

$$\lim_{n\to\infty} \mathcal{L}\Big(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \phi^n(\widehat{\phi}_n - \phi)\Big) = \mathcal{L}\Big(\frac{(\phi^2 - 1)\mathcal{Z}_\infty}{S_\infty + X_0}\Big)$$

where the random variables $\mathcal{Z}_\infty$, S_∞ *and* X_0 *are independent with*

$$\mathcal{L}(\mathcal{Z}_\infty) = \mathcal{N}\big(0, (\phi - 1)^{-2}\varsigma^2\big) \quad \text{and} \quad S_\infty = \sum_{k=1}^{\infty} \phi^{-k} Z_k.$$

When $X_0 = 0$ and the Z_ns are Gaussian, then the law of limit is Cauchy distribution.

References

- P.J. Brockwell and R.A. Davis (1987). *Times Series: Theory and Methods*. Springer.
- H.L. Hurd and A. Miamee (2007). *Periodically Correlated Random Sequences: Spectral Theory and Practice*. Wiley-Interscience.
- M.H. Mehlman. *Stoch. Anal. Appl.* 9 (1991), 323–361.
- M.H. Mehlman. *J. Multivariate Anal.* 43 (1992), 147–170.
- H. Niemi. *J. London Math. Soc.* (2), 17 (1978), 187–192.
- M.M Rao. *Enseign. Math.* 28 (1982), 295–351.
- Yu. Rozanov. *Th. Probability. Appl.* 4 (1959), 271-287.

- N. Dunford and J. T. Schwartz (1957). *Linear Operators, Part I.* Wiley-Interscience.
- W. Rudin (1966). *Real and Complex Analysis*, McGraw-Hill.
- W. Rudin (1967). *Fourier Analysis on Groups*, Wiley.
- B. Sz-Nagy and C Foiaș (1970). *Harmonic Analysis of Operators on Hilbert Space*. North-Holland Publishing Company,

- H.B. Mann and A. Wald. *Econometrica* vol.11. n° 3 & 4 (1943).
- T.W. Anderson. *Ann. Math. Stat.* 30 (1959), 676–687.
- J.S. White. *Ann. Math. Statist.* 29 (1958), 1188–1197.
- P.D. Feigin. *Adv. Appl. Probab.* vol.8. (1976), 712–736.
- P.D. Feigin. *J. Appl. Probab.* vol.16 (1979), 440–444.

THANK YOU FOR YOUR ATTENTION