Notes on the Reproducing Kernel Hilbert Space For Periodically Correlated Processes

by

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Introduction

Suppose $X(t)$, $t \in \mathbb{R}$ is a second order continuous time complex-valued PC process with period T and mean $E\{X(t)\} = 0$. We may then write

$$R_X(u,v) = R_X(u+T, v+T) \tag{1}$$

for every u, v.

The purpose of these notes is to sketch some facts about the reproducing kernel Hilbert space (RKHS) associated with X. We denote $H(X) = \overline{sp}\{X(t), t\in\mathbb{R}\}$ as the Hilbert space of the process X and $H(R_X) = \{f: \mathbb{R}\to\mathbb{C},\ f(t) = E\{\xi\overline{X(t)}\} \text{ for } \xi \in H(X)\}$ is the associated RKHS. The inner product on $H(R_X)$ is defined by

$$<f,g>_{H(R_X)} = E\{\xi\overline{\eta}\} = <\xi,\eta>_{L_2(\Omega,\mathcal{F},P)} \tag{2}$$

where ξ and η correspond to f and g through the definition of $H(R_X)$. It may be shown that

(i) $\overline{R_X}(\cdot,t) \in H(R_X) \quad \forall\ t \in \mathbb{R}$ $\qquad (\xi = X(t))$

(ii) $\text{span}\{\overline{R_X}(\cdot,t)\ t\in\mathbb{R}\}$ is dense in $H(R_X)$ $\qquad \sum \alpha_j \bar{R}_X(\cdot,t_j) = E\{\sum \alpha_j X(t_j)\overline{X(\cdot)}$

(iii) $<f(\cdot), \overline{R_X}(\cdot,t)>_{H(R_X)} = f(t) \quad \forall\ f \in H(R_X)$ and $\forall\ t \in \mathbb{R}$

This is the reproducing property.

(iv) $H(X)$ and $H(R_X)$ are isomorphic with $X(t,\omega) \leftrightarrow \overline{R}(\cdot,t) = R(t,\cdot) \quad \forall\ t \in \mathbb{R}$.

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These elementary properties are stated here for convenience.

We are looking for properties of the elements of $H(R_X)$ that arise from the defining property

(1) of PC processes. We find it helpful to begin with two important sub classes of the PC processes.

Stationary Processes

Since stationary processes are PC with every period, we first look at the form of $H(R_X)$ for stationary processes indexed on the entire real line $\mathbb{R}$. The only feature we currently call attention to arises from property (ii); that is

$$H(R_X) = \overline{sp}\{\overline{R_X}(\cdot,t),\ t\in\mathbb{R}\} = \text{cl}\ \{f(t) = \sum_{j=1}^{N} a_j R_X(t-t_j),\ \text{arbitrary } N,\ t_1 \ldots t_N,\ a_1,\ \ldots,\ a_N\}$$

where cl means closure, in this case with respect to the norm $\|\cdot\|_{H(R_X)}$ induced by $<\cdot,\cdot>_{H(R_X)}$. In words, the elements of $H(R_X)$ are limits of linear combinations of arbitrary translates of $R_X(u)$.

So $f(t) \in H(R_X)$ implies $f(t+\tau) \in H(R_X)$ for every τ; that is, $H(R_X)$ is closed under arbitrary translations. This can be seen in another way. Since X is stationary there exists a one parameter group of unitary operators on H(X) defined by $U^\tau X(t) = X(t+\tau)$. If ξ corresponds (1–1) to $f(\cdot)$, then we write

$$f(t+\tau) = E\{\xi\ \overline{X(t+\tau)}\} = <\xi, U^\tau X(t)>_{H(X)} = <U^{-\tau}\xi,\ X(t)>_{H(X)} \tag{3}$$

so that $f(t+\tau)$ corresponds to $U^{-\tau}\xi$. Further, as we might expect,

$$\|f(t+\tau)\|^2_{H(R_X)} = \|U^{-\tau}\xi\|^2_{H(X)} = \|\xi\|^2_{H(X)} = \|f(t)\|^2_{H(R_X)} \tag{4}$$

so that the length of f(s) in $H(R_X)$ is translation invariant. In fact, $U^{-\tau}$ induces an operator V^τ: $H(R_X) \to H(R_X)$ by $V^\tau f(\cdot) = f(\cdot+\tau)$ and (4) shows that V^τ is unitary and it may be seen that $\{V^\tau, \tau\in\mathbb{R}\}$ is a one parameter group; $V^t \circ V^\tau = V^{t+\tau}$, $(V^\tau)^{-1} = V^{-\tau}$ and $V^0 = I$. From this we may conclude that

$$<f(\cdot+\tau),\ f(\cdot)>_{H(R_X)} = <U^{-\tau}\xi,\ \xi>_{H(X)} = R_\xi(\tau).$$

The same arguments may also be used to show that a one parameter group $\{V^\tau, \tau\in\mathbb{R}\}$ of unitary operators on $H(R_X)$ induces a stationary process on $H(X)$.

Periodic Processes

If there exists a T for which $X(t) = X(t+T)$ (equality in $H(X)$) for every $t \in \mathbb{R}$, we say that X is periodic with period T; we assume that T is the least positive value for which periodicity holds. It is not difficult to characterize the functions in $H(R_X)$.

Proposition

The following three statements are equivalent:

(a) X is periodic with period T

(b) If $f \in H(R_X)$, then $f(t) = f(t+T)$ $\forall$ $t \in \mathbb{R}$ where equality is pointwise.

(c) $R_X(u,v) = R_X(u+mT, v+nT)$ $\forall$ $u,v \in \mathbb{R}$, $m,n \in \mathbb{Z}$ ($R_X(u,v)$ is doubly periodic with period T).

Proof. We show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

To show (a) $\Rightarrow$ (b), let $f \in H(R_X)$ and ξ be the corresponding element in $H(X)$; then

$$|f(t+T) - f(t)| = |<\xi, X(t+T) - X(t)>_{L_2}| \leq \|\xi\|_{L_2}\|X(t+T) - X(t)\|_{L_2} .$$

To show (b) $\Rightarrow$ (c) we note that $\overline{R_X}(\cdot,t) = \overline{R_X}(\cdot+T,t)$ for every t means that $\overline{R_X}(s,t) = \overline{R}(s+T,t)$ for every s,t; from this we conclude that $\overline{R_X}(s,t)$ and hence $R_X(s,t)$ is periodic (with period T) in s (the first variable) for each fixed t. But since also $\overline{R_X}(s,t) = R_X(t,s)$ we conclude that R_X is doubly periodic with period T.

To show (c) $\Rightarrow$ (a), write

$$\|X(t+T) - X(t)\|^2_{L_2} = R_X(t+T, t+T) - R_X(t+T,t) - R_X(t,t+T) + R_X(t,t)$$

and the result immediately follows. □

So periodic processes correspond to $H(R_X)$ consisting only of pointwise periodic functions.

Thus the function $f \in H(R_X)$ may be completely determined by

$$f(t) = \langle \xi, X(t) \rangle_{L_2}, \quad t \in [0,T)$$

and extending f periodically. It is also obviously true (since $f(t) = f(t+T)$) that $f(t+kT)$ is in $H(R_X)$ for any $f \in H(R_X)$ and $\|f(t)\|_{H(R_X)} = \|f(t+kT)\|_{H(R_X)}$; that is $H(R_X)$ is closed under integral T translates. But more important, the integral T translates do not add anything to $H(R_X)$ because of the periodicity.

If X is periodic <u>and</u> stationary then the elements of $H(R_X)$ are periodic and in addition, for all τ, $f(t+\tau) \in H(R_X)$ and $\|f(t)\|_{H(R_X)} = \|f(t+\tau)\|_{H(R_X)}$. In this case $R_X(\tau)$ is periodic in τ with period T and so $\langle f(\cdot+\tau), f(\cdot) \rangle_{H(R_X)}$ will be also.

<u>PC Processes</u>

In general, from property (ii)

$$H(R_X) = \text{cl}\ \{f(t) = \sum_{j=1}^{N} a_j R_X(t,t_j), \text{ arbitrary } N, t_1 \ldots t_N, a_1, \ldots, a_N\}$$

where the closure is in $H(R_X)$. If X is PC, the defining property (1) leads to $R_X(\cdot,t) = R_X(\cdot+T, t+T)$; that is if t is translated by T, the same function of the first variable is obtained, but also translated by T. This produces

$R_X(\cdot, t_j) = R_X(\cdot - T, t_j - T) = R_X(\cdot - k_j T, t_j \bmod$

$$H(R_X) = \text{cl}\ \{f(t) = \sum_{j=1}^{N} a_j R_X(t-k_j T, t_j \bmod T), k_j = [t_j/T], \text{ arbitrary } N, t_1 \ldots t_N\}$$

where [t] denotes the largest integer not exceeding t. If we denote $R' = \{f(t) = R(t,s), s \in [0,T)\}$ then $H(R_X)$ consists of limits of arbitrary integral T translates $g(t) = \Sigma_{j=1}^{N} a_j R_X(t-k_j T, s_j)$ of elements of R'. Said another way, $H(R_X)$ is the smallest subspace containing integral T translates of the family of functions R'. We note the contrast to the stationary case where $H(R_X)$ is the smallest subspace containing arbitrary translates of one function, and the periodic case where all the T translates of R' coincide with R' (no additional functions are obtained by adding the T translates).

The following shows how translations by T are characteristic to PC processes. We begin by examining time translates of length T in H(X). If X(t) is PC with period T, we define a linear operator J: sp(X) $\to$ sp(X) by

$$J\Big(\sum_{j=1}^{N} a_j X(s_j)\Big) = \sum_{j=1}^{N} a_j X(s_j+T)$$

for arbitrary N, $s_1 \ldots s_N$, $a_1, \ldots, a_N$. It is clear from the PC property that if $y_1, y_2 \in$ sp(X) then

$$<y_1, y_2>_{H(X)} = <Jy_1, Jy_2>_{H(X)}$$

so J is a congruence and can be extended to H(X).

Let $t_0 \in \mathbb{R}$ be arbitrary and set

$$H_0(X) = \overline{sp}\{X(t),\ t \in [t_0, t_0+T)\},$$

and generally

$$H_j(X) = \overline{sp}\{X(t),\ t \in [t_0+jT, t_0+(j+1)T)\}.$$

It is clear that the congruence J: $H_j(X) \to H_{j+1}(X)$, so $H_{j+1}(X) = JH_j(X)$ and

$$H(X) \supset \bigcup_j H_j(X) = \bigcup_j J^j H_0(X).$$

Although we cannot claim the reverse inclusion, it is clear that

$$\begin{aligned} H(X) &= \mathrm{cl} \bigcup_j J^j sp\{X(t),\ t \in [t_0+jT, t_0+(j+1)T)\} \\ &= \overline{sp}\ \{\bigcup_j J^j sp\{X(t),\ t \in [t_0+jT, t_0+(j+1)T)\}\} \\ &= \overline{sp}\ \{X(t) : t \in \mathbb{R}\}. \end{aligned}$$

It is not true, in general, that $H_j(X) \perp H_k(X)$ for $j \neq k$, even when X is stationary. This just shows that a PC process is defined by a collection or family of random variables $\{X(s),\ s \in [t_0, t_0+T)\}$ and the orbit of this collection under iterates of a unitary operator J.

We now examine the corresponding action of J in $H(R_X)$. If ξ and $f(\cdot)$ are corresponding elements in $H(X)$ and $H(R_X)$, then

$$f(t) = <\xi, X(t)>_{H(X)} \qquad \text{and define}$$

$$g(t) = <\xi, X(t-T)>_{H(X)} = <\xi, J^{-1}X(t)>_{H(X)} = <J\xi, X(t)>_{H(X)}.$$

Thus $g(t) = f(t-T) \in H(R_X)$ and so we conclude that $H(R_X)$ is closed under integral T translates; but the elements of $H(R_X)$ need not be periodic. Further we note that there is an operator K: $H(R_X) \to H(R_X)$ formed by identifying Kf (with f corresponding to ξ) as the element of $H(R_X)$ corresponding to $J\xi$; so evidently $Kf(\cdot) = f(\cdot-T)$.

For any pair $\xi \leftrightarrow f$ we thus have

$$<K^p f,f>_{H(R_X)} = <f(\cdot-pT), f(\cdot)>_{H(R_X)} = <J^{-p}\xi,\xi>_{H(x)} = R_\xi(p)$$

and $R_\xi(p)$ is a covariance. It is evident that K is unitary and its action on $f(\cdot) \in H(R_X)$ is to shift $f(\cdot)$ to $f(\cdot-T)$.

Now if we denote

$$\tilde{H}_p = \overline{sp}\,\{R_X(\cdot,s),\ s \in [pT, (p+1)T)\},$$

the defining property (1) implies that $\tilde{H}_p$ consists precisely of the $-pT$ translates of $\tilde{H}_0$, that is

$$\tilde{H}_p = \{g(t) = f(t-pT),\ f \in \tilde{H}_0\} = K^p\tilde{H}_0.$$

It is also clear that there is a congruence between $\tilde{H}_p$ and H_p obtained from the congruence between $H(R_X)$ and $H(X)$.

We shall subsequently return to these spaces in the examination of the Karhunen-Loève representation, but we first examine the RKHS for some particular subclasses of PC processes.

Examples of the RKHS for Some Specific PC Processes

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1. Amplitude Scale Transformation or Modulation

Let $X(t)$, $t \in \mathbb{R}$ be wide sense stationary, with $H(X)$ and $H(R_X)$ defined previously. If $P(t)$ is periodic with period T, then $Y(t) = P(t)X(t)$ may be seen to be PC with period T. Clearly $H(Y) \subset H(X)$ where the inclusion may be proper if $P(t)$ has zeros, and so $H(R_Y) \subset H(R_X)$. So if $\xi \in H(Y) \subset H(X)$, then

$$f_Y(t) = <\xi, P(t)X(t)> = \bar{P}(t)<\xi, X(t)> = \bar{P}(t)f_X(t).$$

So the elements of $H(R_Y)$ are products of $\bar{P}(t)$ with $f_X(t) \in H(R_X)$ (recall the nature of $f_X(t)$ for X stationary). We can easily check that $H(R_Y)$ is closed under integral T translates as if $f_Y(t) \in H(R_Y)$, then

$$f_Y(t-T) = \bar{P}(t-T)f_X(t-T) = \bar{P}(t)f_X(t-T) \in H(R_Y)$$

because $f_X(\cdot-T) \in H(R_X)$. Recall $f_X(\cdot+\tau) \in H(R_X)$ for every τ; this, however, does not say that arbitrary translates $f_Y(\cdot+\tau)$ are in $H(R_Y)$ for every τ.

2. Time Scale Transformation

The next example is given by the time-scale transformation(modulation) $Y(t) = X(t+P(t))$ where X and $P(t)$ are as above. If $P(t)$ is continuous (and thus bounded), then $H(Y) = H(X)$ and so if $\xi \in H(Y)$

$$f_Y(t) = <\xi, X(t+P(t))> = f_X(t+P(t)).$$

So whenever $f_X(\cdot) \in H(R_X)$, $f_X(t+P(t) \in H(R_Y)$. Specifically, since X is stationary, $f_X(\cdot-T) \in H(R_X)$ and we may again verify that $H(R_Y)$ is closed under T translates as it must be. Write

$$f_X(t-T+P(t)) = f_X(t-T+P(t-T)) = f_Y(t-T+P(t)) \in H(R_Y).$$

It may be of interest (later) to examine the spaces H_j and $\tilde{H}_j$ for these examples.

3. Harmonizable PC Processes

The final example we consider is when X is harmonizable (strongly) and we may write

$$X(t) = \int_{-\infty}^{\infty} \exp(i\lambda t)\ Z(d\lambda).$$

In this event, every $\xi \in H(X)$ may be written as

$$\xi = \int_{-\infty}^{\infty} g(\lambda)\ Z(d\lambda)$$

where $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\lambda_1)\overline{g(\lambda_2)}\ r_Z(d\lambda_1, d\lambda_2) < \infty$; then

$$f(t) = <\xi, X(t)> = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\lambda_1) \exp(-it\lambda_2)\ r_Z(d\lambda_1, d\lambda_2).$$

For harmonizable PC processes, the support of r_Z is confined to lines $\lambda_2 = \lambda_1 - 2\pi k/T$, and so we may write

$$f(t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} g(\lambda) \exp[-i(\lambda-2\pi k/T)t]\ r_k(d\lambda) = \sum_{k=-\infty}^{\infty} \underline{\exp(i2\pi kt/T)}\ \underline{g_k(t)}$$

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where

$$g_k(t) = \int_{-\infty}^{\infty} \exp(-i\lambda t)\ g(\lambda)\ r_k(d\lambda).$$

For now we ignore the issue of existence of these integrals. The primary point is the form of f(t), namely a sum of products of the form $\exp(i2\pi kt/T)\ g_k(t)$, where the $g_k(t)$ are weighted Fourier transforms of r_k. It is clear that f(t) for the amplitude modulation example can be put into this form if P(t) has a finite Fourier series by writing

$$f_Y(t) = \bar{P}(t)\ f_X(t) = \sum_{k=-N}^{N} \bar{P}_k \exp(-i2\pi kt/T)\ f_X(t)$$

so we identify $g_k(t)$ with $\bar{P}_k f_X(t)$. The general form for f(t) is interesting as it appears as a Fourier series with variable coefficients. Recall when X is a periodic process, $f(t) = \Sigma_k f_k \exp(i2\pi kt/T)$, the coefficients are constant. Further, when X is stationary, f(t) is represented by only the central term

$$f(t) = \langle \xi, X(t) \rangle = \int_{-\infty}^{\infty} g(\lambda) \exp(-i\lambda t)\, r_0(d\lambda)$$

where r_0 is the spectral measure of X. We also note that the harmonizability of X implies f(t) is bounded and continuous.

Interpretation

Since $\xi = \int_{-\infty}^{\infty} g(\lambda)\, Z(d\lambda)$, the function $g(\lambda)$ describes the contribution to ξ of Z in the neighborhood of λ. The measure r_k describes the correlation of increment $Z(d\lambda)$ with increment $Z(d\lambda+2\pi k/T)$, indexed as a function of λ; r_k may be considered a correlation measure for the difference frequency $\lambda_1-\lambda_2 = 2\pi k/T$. Thus $g_k(t) = \int_{-\infty}^{\infty} \exp(-i\lambda t)\, g(\lambda)\, r_k(d\lambda)$ is the g-weighted Fourier transform with respect to the correlation measure r_k. At this time we are unable to give any further intrepretation of the form for f(t).

Implications of the Karhunen-Loève Representation

Let us now restrict our attention to X(t) for $t\in[t_0, t_0+T)$; for convenience we take $t_0 = 0$ but we should keep in mind that t_0 is an implicit parameter in the following development. We also assume that $R_X(s,t)$ is real valued and continuous on $[0,T) \times [0,T)$ and so there exists (Mercer's theorem) a sequence of eigenfunctions $\{\phi_p(t),\ p=1,2,...\}$ and non negative eigenvalues λ_p for which

$$\int_0^T R_X(s,t)\, \phi_p(s) ds = \lambda_p \phi_p(t) \tag{5a}$$

$$\int_0^T \phi_p(t)\, \phi_q(t) dt = \delta_{p,q} \tag{5b}$$

$$R_X(s,t) = \sum_1^{\infty} \lambda_p\, \phi_p(s)\, \phi_q(t) \tag{5c}$$

where the convergence is absolute and uniform on $[0,T) \times [0,T)$. It then follows that there exists a sequence $\{X_{0p},\ p = 1, 2, ...\}$ of orthogonal random variables in H_0 for which

$$<X_{0p}, X_{0q}>_{H(X)} = \lambda_p \delta_{p,q}$$

and for every $t \in [0,T)$,

$$X(t) = \sum_{p=1}^{\infty} \phi_p(t)\, X_{0p}. \tag{6}$$

The first subscript of X_{0p} refers to the "0th" interval $[0,T)$; next we see what happens in the jth interval $[jT, (j+1)T)$.

For $t \in [jT, (j+1)T)$, the defining property (1) implies that the sequence $\{\phi_p(t),\ p=1,2,...\}$ also solves

$$\int_{jT}^{(j+1)T} R_X(s,t)\, \phi_p(s-jT)ds = \lambda_p\, \phi_p(t-jT), \quad jT \leq t < (j+1)T$$

or in other words, if we take $\phi_p(t)$ to be extended periodically, then we can write

$$\int_{jT}^{(j+1)T} R_X(s,t)\, \phi_p(s)ds = \lambda_p\, \phi_p(t), \quad jT \leq t < (j+1)T, \tag{7a}$$

$$\int_{jT}^{(j+1)T} \phi_p(t)\, \phi_q(t)dt = \delta_{p,q} \tag{7b}$$

and

$$R_X(s,t) = \sum_{1}^{\infty} \lambda_p\, \phi_p(s)\, \phi_q(t)\, . \tag{7c}$$

From this it is clear that for any j there is a sequence $\{X_{jp},\ p=1,2,...\}$ of orthogonal random variables in H_k for which

$$<X_{jp}, X_{jq}>_{H(X)} = \lambda_p \delta_{p,q}$$

and for every $t \in [jT, (j+1)T)$

$$X(t) = \sum_{p=1}^{\infty} \phi_p(t)\, X_{jp}\, , \tag{8}$$

where the functions $\{\phi_p(t),\ p=1,2,\ldots\}$ are taken in the extended sense.

We shall now show that $H_j = \overline{sp}\ \{X_{jp},\ p=1,2,\ldots\}$ and that for any j and p, $X_{j+1,p} = JX_{j,p}$. To see these results we need to use the fact that X_{jp} may be obtained from X(t) by

$$X_{jp} = \int_{jT}^{(j+1)T} X(t)\ \phi_p(t)dt \tag{9}$$

where the integral will exist as a Riemann stochastic integral because $\phi_p(t)$ is continuous and X(t) is continuous in quadratic mean. Hence each $X_{jp} \in H_j$ and so using (8) we conclude

$$\{X(t),\ t \in [jT,\ (j+1)T)\} \subset \overline{sp}\{X_{jp},\ p=1,2,\ldots\} \subset H_j\ ;$$

taking the span closure of the leftmost set gives the result that $H_j = \overline{sp}\{X_{jp},\ p=1,2,\ldots\}$. Since we have already established that JX(t) = X(t+T), then from (9)

$$JX_{jp} = J\left[\int_{jT}^{(j+1)T} X(t)\ \phi_p(t)dt\right] = \int_{jT}^{(j+1)T} JX(t)\ \phi_p(t)dt$$

$$= \int_{jT}^{(j+1)T} X(t+T)\ \phi_p(t)dt = X_{j+1,p}\ . \tag{10}$$

The operator J may be brought inside the integral because it is linear and unitary; consider J acting on finite approximations to the integral. A different sequence $\{X_{jp},\ p=1,2,\ldots\}$ of orthogonal random variables is required to represent X(t) on each interval $[jT,\ (j+1)T)$. Since J is unitary, the action of J is to rotate the elements of $\{X_{jp},\ p=1,2,\ldots\}$ while preserving their length and mutual orthogonality.

The relationship (10) expresses the fact that the collection $\{X_{jp},\ p=1,2,\ldots\}$ may be considered on infinite dimensional stationary vector sequence (with time parameter j). We denote $r_{pq}(j-k) = E\{X_{jp}\overline{X}_{kq}\}$ as the infinite dimensional covariance matrix of the sequence $\{X_j\}$. Using (9), we may compute $r_{pq}(j-k)$ by

$$r_{pq}(j-k) = \int_{jT}^{(j+1)T} \int_{kT}^{(k+1)T} R_X(s,t)\ \phi_p(s)\ \phi_q(t)\ dsdt \tag{11}$$

where $\phi_p(\cdot)$ is taken to be periodically extended; note $r_{pq}(0) = \lambda_p \delta_{p,q}$.

The fact that (11) is a function of (j−k) for fixed p,q may also be confirmed directly in (11) from the defining property (1). Also, the covariance $R_X(s,t)$ may be expressed for all s,t, where for specificity $s \in [jT, (j+1)T)$ and $t \in [kT, (k+1)T)$, by

$$R_X(s,t) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} r_{pq}(j-k)\, \phi_p(s)\, \phi_q(t). \tag{12}$$

We shall now examine the form $r_{pq}(n)$ for some specific examples. *First, if X is stationary* then (11) produces

$$r_{pq}(j-k) = \int_{jT}^{(j+1)T} \int_{kT}^{(k+1)T} R_X(s-t)\, \phi_p(s)\, \phi_q(t)\, dsdt \tag{13}$$

which shows that $r_{pq}(j-k)$ still depends on j−k but does not depend on the time origin of the process (recall the parameter t_0). *Next, we permit X to be PC but $H_k \perp H_j$*, or equivalently,

$$<X_{jp}, X_{kp}> = r_{pq}(j-k) = \lambda_p\, \delta_{p-q}\, \delta_{k-j};$$

in this case every new interval of length T adds a countable subspace to the history of the process (each X_{jp} for fixed p is a white noise process). *Finally*, in the other extreme, suppose every new interval of length T adds nothing, so $H_k = H_j$. An example is when X is periodic because then, from (11), $r_{pq}(j-k) = \lambda_p\, \delta_{p-q}$; that is, $r_{pq}(j-k)$ does not depend on j or k. Note X periodic is equivalent to J = I but it is possible to have $H_k = H_{k+1}$ without J = I; for example, J only needs to rotate all the X_{kp} to preserve their lengths and relative angles (i.e., inner products).

Another Associated Hilbert Space

We next introduce another auxiliary Hilbert space that is used in the RKHS theory for processes defined on [0,T], and should therefore be helpful here. Given a sequence of weights $\lambda = \{\lambda_1, \lambda_2, \ldots\}$, denote $H(\lambda)$ as the set of real sequences square summable with weights λ; that is

$$H(X) = \{a = (a_1,a_2,\ldots): \sum_{p=1}^{\infty} a_p^2 \lambda_p < \infty\}.$$

If for sequences $a,b \in H(\lambda)$ we define the inner product $<a,b>_{H(\lambda)} = \Sigma_{p=1}^{\infty} a_p b_p \lambda_p$, then $H(\lambda)$ may be seen to be a Hilbert space. There is a congruence between $H(\lambda)$ and H_j; if $a,b \in H(\lambda)$, then define $Z_j(a)$, $Z_j(b) \in H_j$ by $Z_j(a) = \Sigma_{p=1}^{\infty} a_p X_{jp}$ and similarly for $Z_j(b)$. Then, since

$<X_{jp}, X_{jq}> = \lambda_p \delta_{p-q}$, we have

$$\|Z_j(a)\|_{H_j}^2 = \sum_{p=1}^{\infty} a_p^2 \lambda_p, \qquad \|Z_j(b)\|_{H_j}^2 = \sum_{p=1}^{\infty} b_p^2 \lambda_p,$$

and

$$<Z_j(a), Z_j(b)>_{H_j} = \sum_{p=1}^{\infty} a_p b_p \lambda_p = <a,b>_{H(\lambda)}.$$

This shows that Z_j maps $H(\lambda)$ into H_j and preserves inner products. The mapping is also onto H_j because if $\xi \in H_j$, $\xi = \Sigma_{p=1}^{\infty} \alpha_p X_{jp}$ (the X_{jp}, $p = 1,\ldots$ form an orthogonal basis for H_j) so $\Sigma_{p=1}^{\infty} \alpha_p^2 \lambda_p < \infty$. Hence Z_j is a congruence map (or there is a congruence between H_j and $H(\lambda)$).

We now observe some additional facts about $r_{pq}(j-k)$. If $x \in H_j$ so $x = \Sigma_{p=1}^{\infty} a_p X_{jp}$ and $y \in H_k$, $y = \Sigma_{q=1}^{\infty} b_q X_{jq}$, then

$$<x,y>_{H(X)} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_p b_p r_{pq}(j-k)$$

$$|<x,y>|_{H(X)} \le \|x\|^2 \, \|y\|^2 = \Big(\sum_{p=1}^{\infty} a_p^2 \lambda_p\Big)\Big(\sum_{p=1}^{\infty} b_p^2 \lambda_p\Big).$$

Thus for any j and k, and any $a,b \in H(\lambda)$, the correlation $r_{pq}(j-k)$ must satisfy

$$\Big|\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_p b_p r_{pq}(j-k)\Big| \le \Big(\sum_{p=1}^{\infty} a_p^2 \lambda_p\Big)^{\frac{1}{2}} \Big(\sum_{p=1}^{\infty} b_p^2 \lambda_p\Big)^{\frac{1}{2}}.$$

In addition, $r_{pq}(j-k)$ is non-negative definite in the sense implied by $\|\Sigma_{\mu=1}^{m} \alpha_\mu X_\mu\|^2 \ge 0$ for $X_\mu \in sp\{X_{jp}, j \in \mathbb{Z}, p \in \mathbb{N}\}$. Or equivalently, for arbitrary m, constants $\alpha_1,\ldots,\alpha_m$, sequences a_1, $\ldots,a_m$ ($\in H(\lambda)$) and integers $n_1,\ldots,n_m$

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \alpha_u \overline{\alpha_\nu} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{\mu p}\, a_{\nu q}\, r_{pq}(n_\mu - n_\nu) \geq 0 \,.$$

The RKHS Again

We are now in a position to compute $f(t) = \langle \xi, X(t) \rangle_{H(X)}$ under some restrictions on ξ and the nature of X. First we can compute f(t) if $\xi \in H_j$ and $t \in [kT, (k+1)T)$; for then

$$f(t) = \left\langle \sum_{p=1}^{\infty} a_p X_{jp}, \sum_{q=1}^{\infty} \phi_q(t)\, X_{kq} \right\rangle_{H(X)} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_p\, \phi_q(t)\, r_{pq}(j-k) \tag{14}$$

where we now observe that for any t the sequence $\{\phi_q(t)\} \in H(\lambda)$. Suppose now that $H_k = H_{k+1}$ for all k, then $H_j = H(X)$ and so all $f(t) \in H(R_X)$ are of the form (14). If, in addition, X is periodic, then $r_{pq}(j-k) = \lambda_p\, \delta_{p-q}$, so (14) becomes $f(t) = \Sigma_{p=1}^{\infty} a_p \phi_p(t) \lambda_p$ where $\phi_p(t)$ is interpreted as the periodic extension, which makes f(t) periodic as required.

If $H_j \perp H_k$ for $j \neq k$, then $r_{pq}(j-k) = \lambda_p \delta_{p-q} \delta_{j-k}$ so $f_j(t) = \Sigma_{p=1}^{\infty} a_{jp} \phi_p(t) \lambda_p$ but we must be careful to say for $\xi_j \in H_j$ and $t \in [jT, (j+1)T)$. In this case, set $\xi = \Sigma_{j \in \mathbb{Z}}\, \xi_j$ with $\xi_j \in H_j$; then $\xi_j \perp \xi_k$ implies $\|\xi\|^2 = \Sigma_{j \in \mathbb{Z}} \|\xi_j\|^2$, $f_j(t) \perp f_k(t)$, and

$$f(t) = \sum_{j \in \mathbb{Z}} f_j(t)$$

and

$$\|f\|^2_{H(R_X)} = \sum \|f_j(t)\|^2_{H(R_X)} .$$

Note that $f_j(t) = 0$ for $t \notin [jT, (j+1)T)$ so we can also write $f(t) = f_j(t)$, $t \in [jT, (j+1)T)$.

The general PC case is more difficult because ξ is a limit of linear combinations of $\xi_j \in H_j$. If $\xi = \Sigma_{j \in F} \xi_j$, where the ξ_j no longer need to be mutually orthogonal, then for $t \in [kT, (k+1)T)$

$$f(t) = \left\langle \sum_{j \in F} \sum_{p=1}^{\infty} a_{jp} X_{jp}, \sum_{q=1}^{\infty} \phi_q(t) X_{kp} \right\rangle = \sum_{j \in F} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{jp} \phi_q(t) r_{pq}(j-k).$$

General elements of $H(R_X)$ are limits of linear combinations of this form.