Recent encounters of cyclical long memory

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Longuet-Higgins model: Along x-direction and time t,

$$\zeta(x,t) = \sum_{n=1}^{N} a_n \cos(k_n(\cos\mu_0)x - w_n t + \phi_n)$$

with frequencies w_n, amplitudes $a_n = \sqrt{2S(w_n)\Delta w}$ for spectrum $S(w)$, heading μ_0, random phases ϕ_n, and wave numbers

$$k_n = w_n^2/g. \qquad \text{(Dispersion relation)}$$

At $x = 0$,

$$\boxed{\zeta_0(t) = \sum_{n=1}^{N} a_n \cos(-w_n t + \phi_n)}$$

ACVF:

$$R(h) = \sum_{n=1}^{N} \cos(hw_n) S(w_n)\Delta w \simeq \int_0^\infty \cos(hw)S(w)dw.$$

Wave elevation at non-zero speed

Non-zero forward speed: Setting $x = U_0 t$ for speed U_0, the model becomes

$$\boxed{\zeta_e(t) = \sum\nolimits_{n=1}^{N} a_n \cos(-w_{e,n} t + \phi_n)}$$

for encounter frequencies

$$w_{e,n} = w_n - \frac{U_0}{g}(\cos\mu_0) w_n^2 = w_n - q w_n^2.$$

ACVF:

$$R(h) = \sum_{n=1}^{N} \cos(h w_{e,n}) S(w_n) \Delta w \simeq \int_0^\infty \cos(h w_e) S(w) dw.$$

Original and transformed spectra

Note: With $w_e = w - qw^2 = w - \frac{U_0}{g}\cos\mu_0 w^2$ and for $q > 0$ ($\mu_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$),

$$\int_0^\infty \cos(hw_e)S(w)dw = \int_0^\infty \cos(h\nu)\widetilde{S}(\nu)d\nu,$$

where

$$\widetilde{S}(\nu) = \frac{S(w_1(\nu)) + S(w_2(\nu))}{(1-4q\nu)^{1/2}} + \frac{S(w_3(\nu))}{(1+4q\nu)^{1/2}}$$

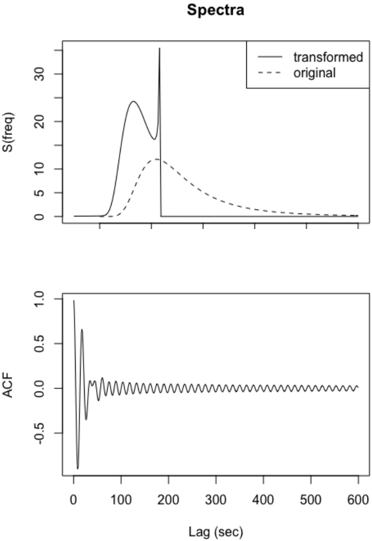
for $\nu \in (0, 1/4q)$, and

$$\widetilde{S}(\nu) = \frac{S(w_3(\nu))}{(1+4q\nu)^{1/2}}$$

for $\nu \in (1/4q, \infty)$.

Power-law divergence of spectrum at non-zero frequency: As $\nu \uparrow 1/4q$ above. (This is directly related to the dispersion relation.)

Original and transformed spectra



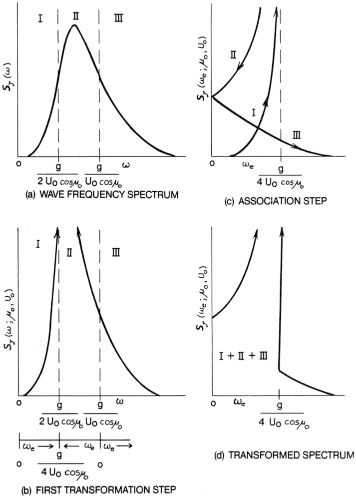
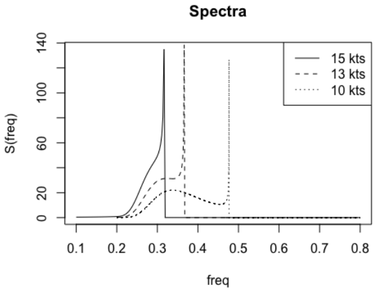
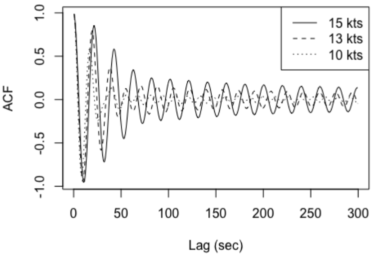


Fig. 68 Transformation of wave spectrum to encounter spectrum, following or quartering waves (long-crested)

Transformed spectrum and ACF

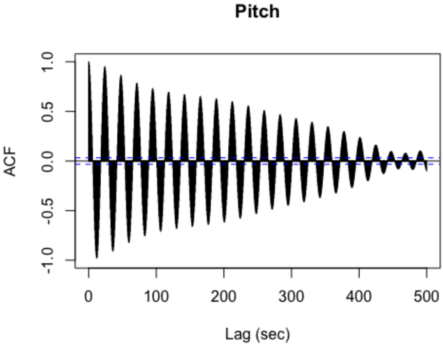


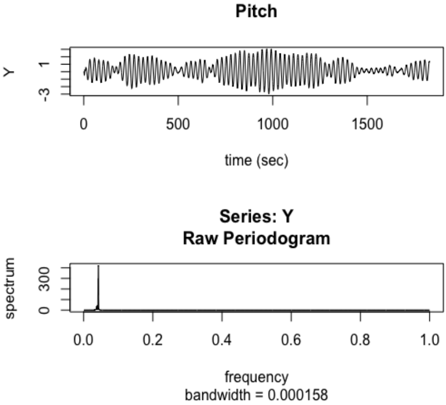


The decay will always be slow but magnitude of ACF coefficients at lags will depend on the underlying spectrum and speed.

Bits of data

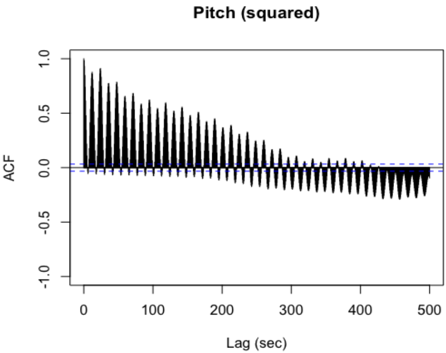
Ship motion: The plot presents one such ACF for the pitch motion from a 30-minute-long record. This is for the flared variant of the ONR Topsides Geometry Series, in sea state 6, the heading of 45°, and traveling at 25 kts.

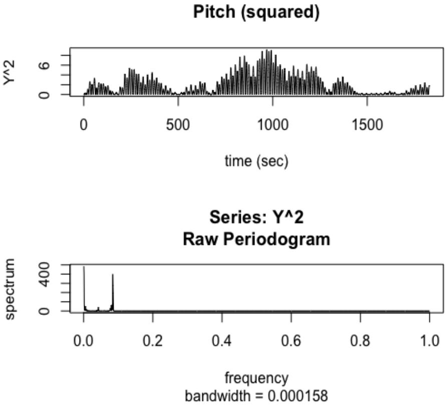




Bits of data

For the same pitch motion process:





How to put confidence intervals, e.g., on the variance?

Cyclical long memory (LM)

Discrete time: Stationary process $X = \{X_n\}_{n\in\mathbb{Z}}$. Assume $\mathbb{E}X_n = 0$.

Spectrum: For some frequency $\nu_0 > 0$, $d \in (0, 1/2)$ and $c_f^+, c_f^- \geq 0$,

$$\boxed{S_X(\nu) \sim \begin{cases} c_f^+(\nu - \nu_0)^{-2d}, & \text{as } \nu \to \nu_0^+, \\ c_f^-(\nu_0 - \nu)^{-2d}, & \text{as } \nu \to \nu_0^-. \end{cases}}$$

ACVF: For $R_X(h) = \mathbb{E}X_{n+h}X_n$,

$$\boxed{R_X(h) \simeq C_R \cos(\nu_0 h + \phi_R) h^{2d-1}, \text{ as } h \to \infty.}$$

Longuet-Higgins: $d = 1/4$, $c_f^+ = 0$, $\phi_R = -\pi/4 \neq 0$.

Notes: $\int_0^\infty |R_X(h)| dh = \infty$, $\left| \int_0^\infty R_X(h) dh \right| < \infty$. Traditional LM: $\nu_0 = 0$.

Cyclical LM

Origins: Hosking (1981), "Fractional differencing," *Biometrika*, last para:

Finally we mention two other processes involving fractional differencing which may prove useful in applications. The fractional equal-root integrated moving-average process is defined by $\nabla^q y_t = (1-\theta B)^q a_t$, $|q| < \frac{1}{2}$, $|\theta| < 1$; as a forecasting model it corresponds to 'fractional order multiple exponential smoothing'. The process $(1-2\phi B + B^2)^d y_t = a_t$, $|d| < \frac{1}{2}$, $|\phi| < 1$, exhibits both long-term persistence and quasiperiodic behaviour; its correlation function resembles a hyperbolically damped sine wave.

There were a number of follow-up papers looking at this phenomenon, with some applications. But it has largely stayed at the margins of LM research. E.g. pp. 185-191 in Giraitis et al. (2012; 500+ pages); pp. 496-499 in Beran et al. (2013; 800+ pages); no mention in Pipiras and Taqqu (2017; 600+ pages); etc.

No other known physical model leading to this phenomenon?

Random modulation viewpoint

Random modulation: Take two independent copies $\{Y_{1,n}\}_{n\in\mathbb{Z}}, \{Y_{2,n}\}_{n\in\mathbb{Z}}$ of LM series Y satisfying $R_Y(h) \sim c_R h^{2d-1}$, as $h \to \infty$. Set

$$X_n = \cos(\nu_0 n) Y_{1,n} + \sin(\nu_0 n) Y_{2,n}, \quad n \in \mathbb{Z}.$$

By construction,

$$\begin{aligned} \mathbb{E} X_{n+h} X_n &= \big(\cos(\nu_0 h)\cos(\nu_0(n+h)) + \sin(\nu_0 h)\sin(\nu_0(n+h))\big) R_Y(h) \\ &= \cos(\nu_0 h) R_Y(h) \sim c_R \cos(\nu_0 h) h^{2d-1}, \quad \text{as } h \to \infty, \end{aligned}$$

and hence $\{X_n\}$ has cyclical LM with $\phi = 0$.

General ϕ: Need to take LM series $\{Y_{1,n}\}_{n\in\mathbb{Z}}$ and $\{Y_{2,n}\}_{n\in\mathbb{Z}}$ dependent in particular way, from both LM perspective and also for all lags. Parametric families can also be constructed with explicit ACVFs.[1]

[1] S. Kechagias, V. Pipiras and P. Zoubouloglou (2024), "Cyclical long memory: Decoupling, modulation and modeling," SPA, 175, 104403.

Conclusions

Key takeaway:

- Model for wave elevation and ship motions at non-zero speed characterized by (cyclical) LM.
- Random modulation sheds light on the nature of cyclical LM.

Question: What is going on with another motion, so-called roll?