Fractional lower-order covariance-based autodependence measures for heavy-tailed cyclostationary time series

Wojciech Żuławiński, Agnieszka Wyłomańska

Wroclaw University of Science and Technology, Poland

Chapel Hill, 14 March 2025

Outline

- Introduction
- Fractional lower-order covariance (FLOC)
- FLOC-cyclostationary time series
- Autodependence measures for FLOC-cyclostationary time series (peFLOACF, peFLOPACF)
- Testing of dependence for FLOC-cyclostationary time series
- Order identification for infinite-variance PAR and PMA models
- Air pollution data analysis
- Summary

This presentation is based on the paper:

[1] W. Żuławiński, A. Wyłomańska, "Fractional lower-order covariance-based measures for cyclostationary time series with heavy-tailed distributions: application to dependence testing and model order identification", submitted to Digital Signal Processing, 2025

Introduction

- For infinite-variance processes, the classical definition of periodically correlated (PC) time series cannot be used.
- We consider a generalization of PC property, where covariance is replaced by fractional lower-order covariance (FLOC).
- This property can be useful for modeling phenomena that exhibit both periodic and heavy-tailed behavior.
- For the analysis of such time series, we propose two new autodependence measures – peFLOACF and peFLOPACF.
- We also present two applications of these measures: for dependence testing and model order identification.
- To further illustrate the practical usefulness of this methodology, real data analysis example is included.

Fractional lower-order covariance (FLOC)

Fractional lower-order covariance (FLOC): alternative to covariance

Definition (Fractional lower-order covariance, FLOC)

For random variables Y_1, Y_2 with finite moments up to some order $1 < a \leq 2$ ($\mathbb{E}|Y_1|^r < \infty$, $\mathbb{E}|Y_2|^r < \infty$ for $0 < r < a$), we define [2]

$$\mathrm{FLOC}(Y_1, Y_2; A, B) = \mathbb{E}[Y_1^{<A>} Y_2^{<B>}],$$

where $x^{<c>} = |x|^c \operatorname{sgn} x$, and $A > 0$, $B > 0$, $A + B < a$.

Fractional lower-order moment (FLOM): alternative to variance

$$\mathrm{FLOM}(Y_1; A, B) = \mathrm{FLOC}(Y_1, Y_1; A, B) = \mathbb{E}|Y_1|^{A+B}.$$

Example of infinite-variance dist. – symmetric α-stable distribution with stability index $\alpha \in (0, 2]$ and scale parameter $\sigma > 0$:

$$X \sim S\alpha S(\alpha, \sigma), \quad \mathbb{E}\exp(izX) = \exp(-\sigma^\alpha |z|^\alpha).$$

FLOC-cyclostationary time series

Recall the classical definition of PC time series.

Definition (Periodically correlated (PC) time series)

A finite-variance time series $\{X_t\}$, $t \in \mathbb{Z}$, is PC if for all $t, h \in \mathbb{Z}$

$$\mathbb{E}X_t = \mathbb{E}X_{t+T}, \quad \mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(X_{t+T}, X_{t+h+T}).$$

To define an analogue of the PC property for infinite-variance case, we replace the autocovariance with its FLOC-based counterpart.

Definition (FLOC-cyclostationary time series)

A time series $\{X_t\}$ (for which given FLOC exists) is said to be FLOC-cyclostationary, if for all $t, h \in \mathbb{Z}$

$$\mathbb{E}X_t = \mathbb{E}X_{t+T}, \;\; \mathrm{FLOC}(X_t, X_{t+h}; A, B) = \mathrm{FLOC}(X_{t+T}, X_{t+h+T}; A, B).$$

Periodic fractional lower-order white noise (peFLOWN)

A basic example of PC time series is periodic white noise (peWN).

Definition (Periodic white noise, peWN)

A finite-variance time series $\{X_t\}$ is peWN with period T if for each $t \in \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$

$$\mathbb{E}X_t = 0, \quad \mathrm{Var}(X_t) = \mathrm{Var}(X_{t+T}), \quad \mathrm{Cov}(X_t, X_{t+h}) = 0.$$

Analogously, as a basic FLOC-cyclostationary time series, we can define the periodic fractional lower-order white noise (peFLOWN).

Definition (Periodic fractional lower-order white noise, peFLOWN)

A time series $\{X_t\}$ (for which given FLOC/FLOM exist) is said to be peFLOWN with period T if for each $t \in \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$

$$\mathbb{E}X_t = 0, \quad \mathrm{FLOM}(X_t; A, B) = \mathrm{FLOM}(X_{t+T}; A, B),$$

$$\mathrm{FLOC}(X_t, X_{t+h}; A, B) = 0.$$

PAR and PMA models

Definition (Periodic autoregressive (PAR) model)

A time series $\{X_t\}$ is $\mathrm{PAR}_T(p)$ if for every $t \in \mathbb{Z}$ we have

$$X_t - \phi_1(t)X_{t-1} - \ldots - \phi_p(t)X_{t-p} = \xi_t,$$

where $\{\xi_t\}$ is peFLOWN, and coefficients $\{\phi_i(t)\}$ are T-periodic in t.

Definition (Periodic moving average (PMA) model)

A time series $\{X_t\}$ is $\mathrm{PMA}_T(q)$ if for every $t \in \mathbb{Z}$ we have

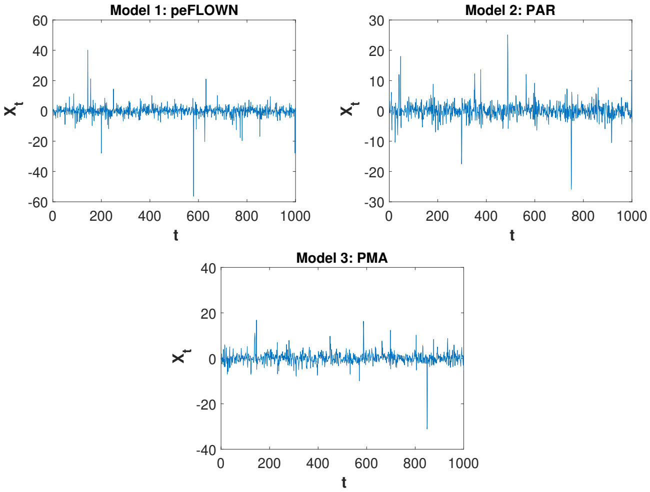
$$X_t = \xi_t + \theta_1(t)\xi_{t-1} + \ldots + \theta_q(t)\xi_{t-q}.$$

where $\{\xi_t\}$ is peFLOWN, and coefficients $\{\theta_j(t)\}$ are T-periodic in t.

$p(t), q(t)$ – "local" orders (i.e., "actual" orders at time t, also T-periodic)

$$p(t) = \max\{i \;:\; i = 1, \ldots, p,\ \phi_i(t) \neq 0\}, \quad q(t) = \max\{j \;:\; j = 1, \ldots, q,\ \theta_j(t) \neq 0\}.$$

FLOC-cyclostationary time series



Sample trajectories of selected FLOC-cyclostationary models with period $T = 2$.
Model 1: peFLOWN, Model 2: $\mathrm{PAR}_2(1)$, Model 3: $\mathrm{PMA}_2(1)$.
(all based on $S\alpha S(1.7, 1)$ distribution)

Digression: autodependence measures for PC processes

For a zero-mean PC time series $\{X_t\}$, the periodic autocovariance function (peACVF) is given by

$$\gamma_v(h) = \mathrm{Cov}(X_{nT+v}, X_{nT+v-h}) = \mathbb{E}[X_{nT+v}X_{nT+v-h}], \quad n \in \mathbb{Z}.$$

One can also consider the normalized version of peACVF, that is, the periodic autocorrelation function (peACF) given by [3]

$$\rho_v(h) = \mathrm{Cov}\left(\frac{X_{nT+v}}{\sqrt{\mathrm{Var}(X_{nT+v})}}, \frac{X_{nT+v-h}}{\sqrt{\mathrm{Var}(X_{nT+v-h})}}\right) = \frac{\gamma_v(h)}{\sqrt{\gamma_v(0)\gamma_{v-h}(0)}}.$$

In other words, $\rho_v(h)$ is the covariance between X_{nT+v} and X_{nT+v-h} standardized to have unit variance.

Digression: autodependence measures for PC processes

Another dependence measure considered for PC processes is the periodic partial autocorrelation function (pePACF).

For a zero-mean PC time series $\{X_t\}$, pePACF $\beta_v(h)$ is defined as the last component of the vector $\phi_{v,h}$ given by [4]

$$\phi_{v,h} = (\mathbf{R}_{v,h})^{-1} \boldsymbol{\rho}_{v,h},$$

where $\mathbf{R}_{v,h}$ (assumed to be non-singular) is $h \times h$ matrix defined as

$$(\mathbf{R}_{v,h})_{i,j} = \rho_{v-j}(i-j), \quad i,j = 1,\dots,h,$$

and

$$\boldsymbol{\rho}_{v,h} = [\rho_v(1), \cdots, \rho_v(h)]'.$$

note: in the literature, there are several conventions of defining pePACF

New autodependence measure: peFLOACF

For FLOC-cyclostationary $\{X_t\}$, we first define the periodic fractional lower order autocovariance function (peFLOACVF) as [5]

$$\psi_v(h) = \mathrm{FLOC}(X_{nT+v}, X_{nT+v-h}; A, B) = \mathbb{E}[X_{nT+v}^{<A>} X_{nT+v-h}^{<B>}].$$

which is a FLOC-based counterpart of peACVF.

Then, we define its normalized version – periodic fractional lower-order autocorrelation function (peFLOACF) – as FLOC between X_{nT+v} and X_{nT+v-h} standardized to have unit FLOM.

The peFLOACF is a FLOC-based counterpart of peACF.

New autodependence measure: peFLOACF

For a FLOC-cyclostationary $\{X_t\}$, peFLOACF is defined as

$$\eta_v(h) = \text{FLOC}\left(\frac{X_{nT+v}}{s(v)}, \frac{X_{nT+v-h}}{s(v-h)}; A, B\right) = \frac{\psi_v(h)}{\psi_v(0)^{\frac{A}{A+B}}\psi_{v-h}(0)^{\frac{B}{A+B}}},$$

where $s(t) = (\psi_t(0))^{\frac{1}{A+B}}$ is the standardizing factor for given X_t.

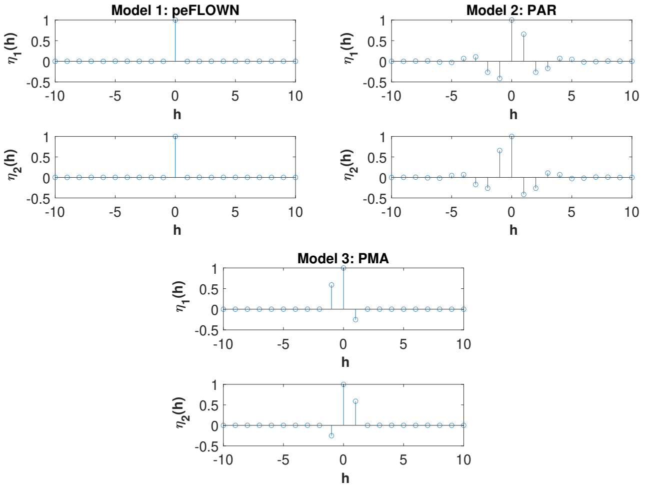
If $A = B$, then the peFLOACF simplifies to

$$\eta_v(h) = \frac{\psi_v(h)}{\sqrt{\psi_v(0)\psi_{v-h}(0)}}.$$

Note: peFLOACF has a cut-off property for $\text{PMA}_T(q)$ models

$$\eta_v(h) = 0 \quad \text{for } |h| > q(v).$$

New autodependence measure: peFLOACF



PeFLOACF $\eta_v(h)$ values ($A = B = 0.8$) for selected FLOC-cyclostationary models with period $T = 2$.
Model 1: peFLOWN, Model 2: $PAR_2(1)$, Model 3: $PMA_2(1)$.

New autodependence measure: peFLOPACF

The periodic fractional lower-order partial autocorrelation function (peFLOPACF) is a FLOC-based counterpart of pePACF.

For FLOC-cyclostationary $\{X_t\}$, the peFLOPACF $\zeta_v(h)$ is defined as the last component of the vector $\phi_{v,h}$ given by $(A = 1)$

$$\phi_{v,h} = \left(\mathbf{H}_{v,h}\right)^{-1} \boldsymbol{\eta}_{v,h},$$

where $\mathbf{H}_{v,h}$ (assumed to be non-singular) is of elements

$$(\mathbf{H}_{v,h})_{i,j} = \eta_{v-j}(i-j), \quad i,j = 1,\ldots,h,$$

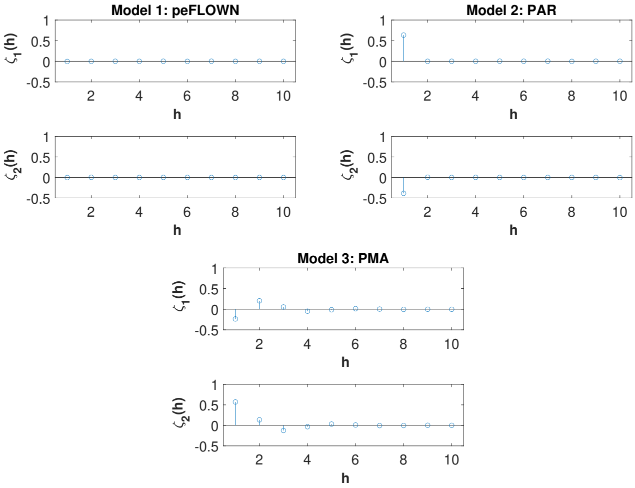
and

$$\boldsymbol{\eta}_{v,h} = [\eta_v(1), \cdots, \eta_v(h)]'.$$

Note: peFLOPACF has a cut-off property for $\mathrm{PAR}_T(p)$ models

$$\zeta_v(h) = 0 \quad \text{for } h > p(v).$$

New autodependence measure: peFLOPACF



PeFLOPACF $\zeta_v(h)$ values ($B = 0.6$) for selected FLOC-cyclostationary models with period $T = 2$.
Model 1: peFLOWN, Model 2: $\text{PAR}_2(1)$, Model 3: $\text{PMA}_2(1)$.

Sample peFLOACF and peFLOPACF

For a sample $x_1, \ldots, x_{NT}$, the sample peFLOACVF is defined as

$$\hat{\psi}_v(h) = \frac{1}{N} \sum_{n=l_b}^{r_b} x_{nT+v}^{<A>} x_{nT+v-h}^{<B>}.$$

Then, the sample peFLOACF is given by

$$\hat{\eta}_v(h) = \frac{\hat{\psi}_v(h)}{\hat{\psi}_v(0)^{\frac{A}{A+B}} \hat{\psi}_{v-h}(0)^{\frac{B}{A+B}}}.$$

The sample peFLOPACF $\hat{\zeta}_v(h)$ is constructed by replacing all terms $\eta_v(h)$ in the peFLOPACF definition with corresponding $\hat{\eta}_v(h)$.

note: l_b, r_b – sum bounds for which the indices $nT + v$ and $nT + v - h$ are always between 1 and NT

Testing of dependence for FLOC-cyclostationary time series

Using peFLOACF, we design a portmanteau test for detecting dependence in FLOC-cyclostationary time series $\{X_t\}$ (adapting the peACF-based test for PC time series [6]).

$$\mathcal{H}_0 : \{X_t\} \text{ is peFLOWN}, \quad \mathcal{H}_1 : \{X_t\} \text{ is not peFLOWN}.$$

We perform a 'subtest' for each $v = 1, \ldots, T$, with the following 'subtest' statistic for given v

$$\kappa_v = N \sum_{h \in H_\pm} (\hat{\eta}_v(h))^2,$$

where $H_\pm = \{-h_{\max}, \ldots, h_{\max}\} \setminus \{0\}$, for some assumed $h_{\max}$.

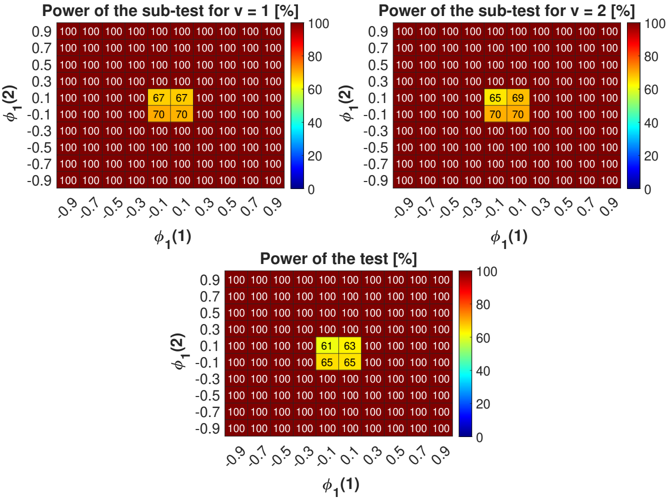
If for any v the value of κ_v is "atypically large", dependence is present in the analyzed series.

Testing of dependence for FLOC-cyclostationary time series

Procedure for dependence testing in sample $x_1, \ldots, x_{NT}$ that corresponds to a FLOC-cyclostationary and $S\alpha S$ time series $\{X_t\}$, for assumed α and significance level c:

- Find the critical region for each subtest (assuming $v = 1$):
  - Generate M i.i.d. sequences of length NT from $S\alpha S(\alpha, 1)$.
  - For i-th trajectory, calculate κ_v denoted as $\kappa_v^{(i)}$.
  - Construct the critical region $(Q_{1-\tilde{c}}, \infty)$, where $\tilde{c} = c/T$, and $Q_{1-\tilde{c}}$ is the quantile of order $1 - \tilde{c}$ of $[\kappa_v^{(1)}, \ldots, \kappa_v^{(M)}]$.
- For each $v = 1, \ldots, T$, perform the subtest:
  - From $x_1, \ldots, x_{NT}$, calculate κ_v denoted as $\kappa_v^{(0)}$.
  - If $\kappa_v^{(0)} \in (Q_{1-\tilde{c}}, \infty)$, $\mathcal{H}_0$ is rejected.

Testing of dependence for FLOC-cyclostationary time series



Empirical powers of both subtests and the entire portmanteau test for different values of $\phi_1(1)$ and $\phi_1(2)$ in $\mathrm{PAR}_2(1)$ model in alternative hypothesis $\mathcal{H}_1$, calculated for trajectories of length $NT = 1000$.

Order identification for infinite-variance PAR model

Using the cut-off property of peFLOPACF, we design a procedure for infinite-variance PAR model order identification.
(adapting the pePACF-based method for finite-variance PAR [7])

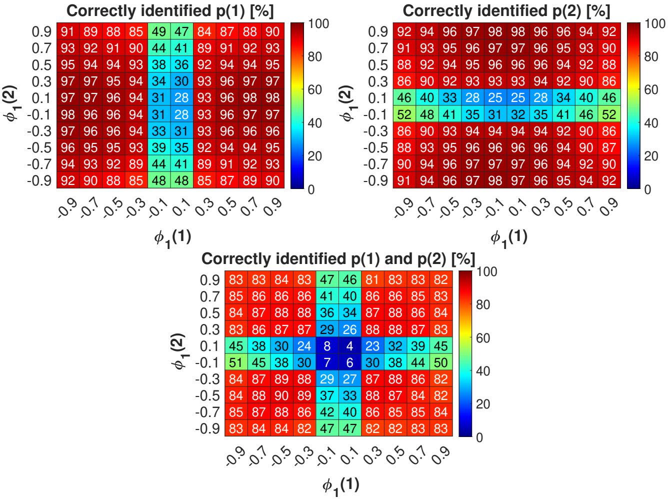
For each $v = 1, \ldots, T$, we look for the largest argument h for which the sample peFLOPACF $\hat{\zeta}_v(h)$ is "significantly non-zero"*.

This argument is then the identified "local" order $p(v)$ for given v.

The "global" order p can then be derived as the maximum of all seasonal orders $p(1), \ldots, p(T)$.

* according to the confidence interval for $\hat{\zeta}_v(h)$ for peFLOWN series obtained using Monte Carlo simulations

Order identification for infinite-variance PAR model



Percentage of cases with correctly identified $p(1)$, $p(2)$ and both $p(1), p(2)$ for different values of $\phi_1(1)$ and $\phi_1(2)$ in $\mathrm{PAR}_2(1)$ model, for trajectories of length $NT = 1000$.

Order identification for infinite-variance PMA model

Using the cut-off property of peFLOACF, we design a procedure for infinite-variance PMA model order identification. (adapting the peACF-based method for finite-variance PMA [3])

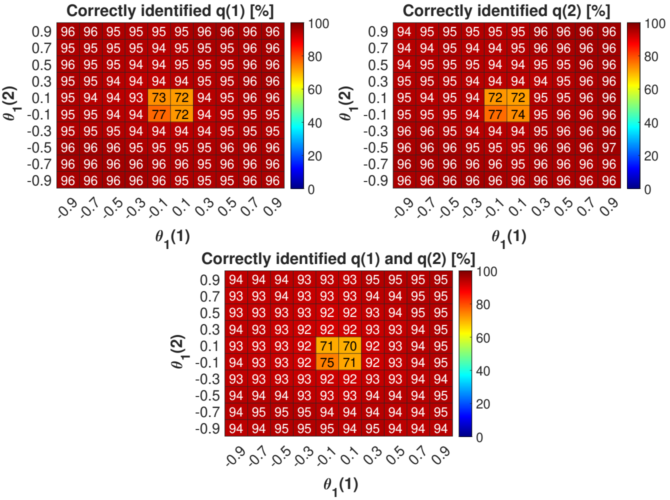
For each $v = 1, \ldots, T$, we look for the smallest non-negative integer k for which the sample peFLOACF $\hat{\eta}_v(h)$ is "close to zero"* for all $h < -k$ and $h > k$.

This argument is then the identified "local" order $q(v)$ for given v.

The "global" order q can then be derived as the maximum of all seasonal orders $q(1), \ldots, q(T)$.

* according to the confidence interval for $\hat{\eta}_v(h)$ for peFLOWN series obtained using Monte Carlo simulations

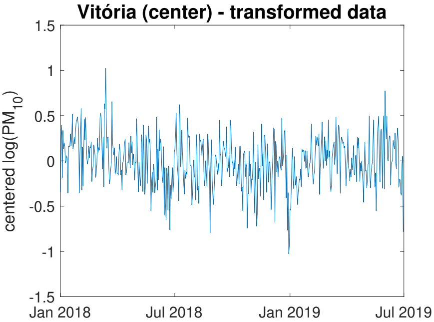
Order identification for infinite-variance PMA model



Percentage of cases with correctly identified $q(1)$, $q(2)$ and both $q(1), q(2)$ for different values of $\theta_1(1)$ and $\theta_1(2)$ in $\mathrm{PMA}_2(1)$ model, for trajectories of length $NT = 1000$.

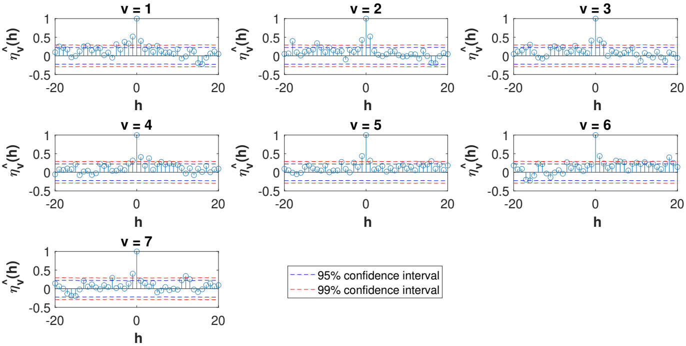
Air pollution data analysis

Analyzed data: preprocessed daily average PM_{10} in Vitória, Brazil.



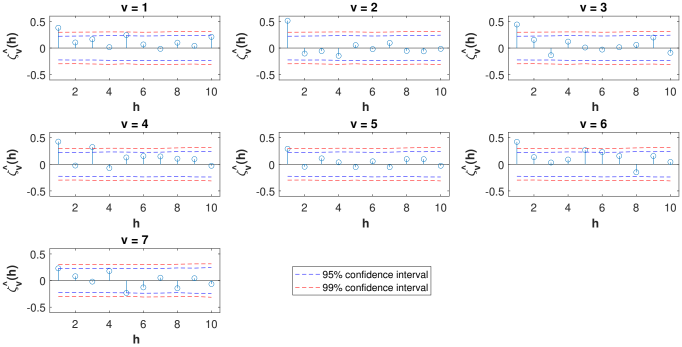
Assumptions: $T = 7$ (weekly rhythm), $\alpha = 1.9$.

Air pollution data analysis



Sample peFLOACF $\hat{\eta}_v(h)$ (with $A = B = 0.85$) for $v = 1, \ldots, T$ for the analyzed dataset with confidence intervals at 95% and 99% levels.

Air pollution data analysis



Sample peFLOPACF $\hat{\zeta}_v(h)$ (with $B = 0.7$) for $v = 1, \ldots, T$ for the analyzed dataset with confidence intervals at 95% and 99% levels.

Air pollution data analysis

Subtest statistic values of the portmanteau test for the dataset:

v	1	2	3	4	5	6	7
κ_v	864.3	648.4	630.0	607.7	562.1	905.3	436.4
critical region			$(458.9, \infty)$				

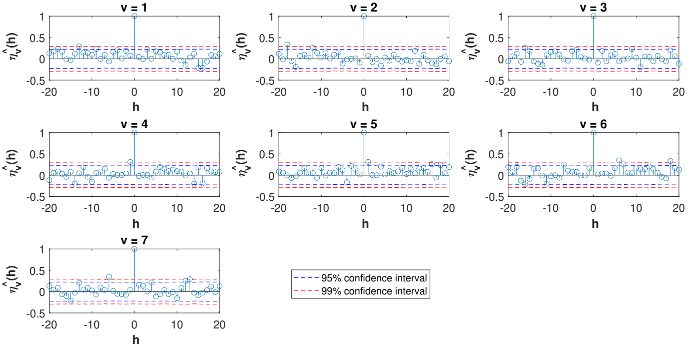
We fit the α-stable $\text{PAR}_7(3)$ model to this dataset (with order identified using the proposed method).

Subtest statistic values of the portmanteau test for the residuals:

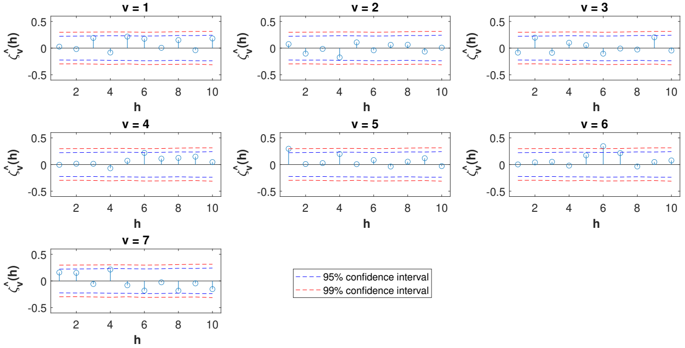
v	1	2	3	4	5	6	7
κ_v	403.3	254.6	292.7	287.1	354.9	450.2	320.6
critical region			$(458.9, \infty)$				

Stability index value estimated from residuals: $\alpha = 1.89$.

Air pollution data analysis



Sample peFLOACF $\hat{\eta}_v(h)$ (with $A = B = 0.85$) for $v = 1, \ldots, T$ for the residuals of the fitted PAR model with confidence intervals at 95% and 99% levels.



Sample peFLOPACF $\hat{\zeta}_v(h)$ (with $B = 0.7$) for $v = 1, \ldots, T$ for the residuals of the fitted PAR model with confidence intervals at 95% and 99% levels.

Summary

- In this presentation, the class of FLOC-cyclostationary time series (generalization of PC time series) was analyzed.
- For these processes (which may have infinite variance), the periodic structure is described using FLOC measure.
- For the analysis of FLOC-cyclostationary processes, the peFLOACF and peFLOPACF measures were introduced.
- Moreover, using these measures, the procedures for dependence testing and order identification were designed.
- The applications to simulated and real data indicate that the proposed methodology is efficient and useful in practice.

Acknowledgements

The work was supported by National Center of Science under Sheng2 grant No. UMO-2021/40/Q/ST8/00024 "NonGauMech - New methods of processing non-stationary signals (identification, segmentation, extraction, modeling) with non-Gaussian characteristics for the purpose of monitoring complex mechanical structures".

The authors thank Prof. Valderio Reisen for sharing the analyzed air pollution data.

References

- **[1] W. Żuławiński, A. Wyłomańska, "Fractional lower-order covariance-based measures for cyclostationary time series with heavy-tailed distributions: application to dependence testing and model order identification", submitted to Digital Signal Processing, 2025**
- [2] X. Ma and C. L. Nikias, "Joint estimation of time delay and frequency delay in impulsive noise using fractional lower order statistics," IEEE Trans. Signal Process. 44 (1996), 2669–2687
- [3] T. A. Ula, A. A. Smadi, "Identification of periodic moving-average models," Commun. Stat. Theory Methods 32 (2003) 2465–2475
- [4] M. S. Mondal, S. A. Wasimi, Generating and forecasting monthly flows of the Ganges river with PAR model, J. Hydrol. 323 (2006) 41–56.
- [5] W. Żuławiński, P. Kruczek, A. Wyłomańska, Alternative dependency measures-based approach for estimation of the α-stable periodic autoregressive model, Commun. Stat. Simul. Comput. 53 (3) (2024) 1188–1215
- [6] A. I. McLeod, Diagnostic checking of periodic autoregression models with application, J. Time Ser. Anal. 15 (2) (1994) 221–233.
- [7] K. W. Hipel, A. I. McLeod, Time Series Modeling of Water Resources and Environmental Systems, Elsevier, 1994.

Thank you for your attention!